Lecture 2: Tilting

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Let $p$ be a prime number, which we regard as fixed throughout this lecture. In Lecture 1, we defined the \textit{tilt} $K^\flat$ of an algebraically closed completely valued field $K$ of residue characteristic $p$. In this lecture, we review the tilting construction in more detail, working in the more general setting of \textit{perfectoid fields}.

**Definition 1.** A \textit{perfectoid field} is a field $K$ equipped with a nonarchimedean absolute value $||_K : K \to \mathbb{R}_{\geq 0}$ satisfying the following axioms:

(A1) The residue field $k = O_K/m_K$ has characteristic $p$. Equivalently, the prime number $p$ belongs to the maximal ideal $m_K$, so that $|p|_K < 1$.

(A2) The field $K$ is complete with respect to the absolute value $||_K$.

(A3) The Frobenius map $\varphi : O_K/pO_K \to O_K/pO_K$ is surjective. That is, for every element $x \in O_K$, we can write $x = y^p + pz$ for some $y, z \in O_K$.

(A4) The maximal ideal $m_K$ is not generated by $p$. In other words, there exists some element $x \in K$ satisfying $|p|_K < |x|_K < 1$.

**Remark 2.** In the situation of Definition 1, choose $x \in K$ satisfying $|p|_K < |x|_K < 1$. Then $x \in O_K$, so we can write $x = y^p + pz$ for some $y, z \in O_K$. Since $|pz|_K \leq |p|_K < |x|_K$, we must have $|x|_K = |y^p|_K = |y|_K^p$. In particular, we have $|x|_K < |y|_K < 1$, so that $y \in O_K \setminus xO_K$. It follows that the maximal ideal $m_K$ is not principal: that is, the valuation ring $O_K$ is not a discrete valuation ring.

**Remark 3.** In the situation of Definition 1, suppose that $K$ is characteristic $p$. In this case, axiom (A1) is automatic, axiom (A3) says that the field $K$ is \textit{perfect} (that is, every element of $K$ has a $p$th root), and axiom (A4) says that the absolute value $||_K$ is nontrivial. In other words, a perfectoid field of characteristic $p$ is just a completely valued perfect field of characteristic $p$.

**Example 4.** Let $K$ be a completely valued field of residue characteristic $p$. Suppose that every element $x \in K$ has a $p$th root (this condition is satisfied, for example, if $K$ is algebraically closed). Then axioms (A3) and (A4) are satisfied, so $K$ is a perfectoid field.

**Example 5.** For each $n > 0$, let $\mathbb{Z}[\zeta_{p^n}]$ denote ring obtained from $\mathbb{Z}$ by adjoining a primitive $p^n$th root of unity, given by the quotient $\mathbb{Z}[x]/(1 + x^{p^{n+1}} + x^{2p^{n+1}} + \cdots + x^{(p-1)p^{n+1}})$; equivalently $\mathbb{Z}[\zeta_{p^n}]$ can be described as the ring of integers in the number field $\mathbb{Q}(\zeta_{p^n})$.

Let $\mathbb{Z}_p^{\text{cyc}}$ denote the $p$-adic completion of the union $\bigcup_{n>0} \mathbb{Z}[\zeta_{p^n}]$ and set $\mathbb{Q}_p^{\text{cyc}} = \mathbb{Z}_p^{\text{cyc}}[1/p]$. Then $K = \mathbb{Q}_p^{\text{cyc}}$ is a perfectoid field with ring of integers $O_K = \mathbb{Z}_p^{\text{cyc}}$. Axiom (A3) follows from the observation that the image of the Frobenius map

$\varphi : \mathbb{Z}_p^{\text{cyc}}/p\mathbb{Z}_p^{\text{cyc}} \to \mathbb{Z}_p^{\text{cyc}}/p\mathbb{Z}_p^{\text{cyc}}$

is a subgroup of $\mathbb{Z}_p^{\text{cyc}}/p\mathbb{Z}_p^{\text{cyc}} \simeq \bigcup_{n>0} \mathbb{F}_p[\zeta_{p^n}]$ which contains each of the roots of unity $\zeta_{p^n}$, by virtue of the equation $\zeta_{p^n} = (\zeta_{p^{n+1}})^p$.

Note that the $p$th power map $\mathbb{Q}_p^{\text{cyc}} \to \mathbb{Q}_p^{\text{cyc}}$ is not surjective: for example, there is no element $x \in \mathbb{Q}_p^{\text{cyc}}$ satisfying $x^p = p$. 

As in the previous lecture, we let $K^\flat$ denote the inverse limit of the system

$$\cdots \to K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K,$$

whose elements can be identified with sequences $\vec{x} = \{x_0, x_1, \ldots \in K : x_n = x_{n+1}^p\}$. We regard $K^\flat$ as a monoid with respect to the obvious multiplication

$$\{x_n\}_{n \geq 0} \cdot \{y_n\}_{n \geq 0} = \{x_n \cdot y_n\}_{n \geq 0}.$$

When $K$ is a perfectoid field, we can equip $K^\flat$ with a compatible addition law. To prove this, it is convenient to first work with the subset $O^\flat_K \subseteq K^\flat$ consisting of those sequences $\{x_n\}_{n \geq 0}$ where each $x_n$ belongs to $O_K$ (note that if this condition is satisfied for any integer $n \geq 0$, then it is satisfied for all integers $n \geq 0$).

**Proposition 6.** Let $K$ be a completely valued field of residue characteristic $p$. Then canonical map $O_K \to O_K/pO_K$ induces a bijection

$$O_K^\flat \to \varprojlim(\cdots \to O_K/pO_K \xrightarrow{x \mapsto x^p} O_K/pO_K).$$

**Proof.** Let us assume that $K$ has characteristic zero (in characteristic $p$, there is nothing to prove). Our assumption that $K$ is complete implies that $O_K$ can be realized as the inverse limit $\varprojlim_n O_K/p^nO_K$. For each $n \geq 1$, let $Z(n)$ denote the limit of the inverse system of sets

$$\cdots \to O_K/p^nO_K \xrightarrow{x \mapsto x^p} O_K/p^nO_K \xrightarrow{x \mapsto x^p} O_K/p^nO_K \xrightarrow{x \mapsto x^p} O_K/p^nO_K.$$

Then $O_K^\flat$ is the inverse limit $\varprojlim_n Z(n)$, and we wish to show that the projection map $O_K^\flat \to Z(1)$ is a bijection. For this, it will suffice to show that each of the transition maps $Z(n) \to Z(n-1)$ is a bijection. In other words, it will suffice to show that the vertical maps in the diagram

$$\cdots \to O_K/p^nO_K \xrightarrow{(\cdot)^p} O_K/p^nO_K \xrightarrow{(\cdot)^p} O_K/p^nO_K \xrightarrow{(\cdot)^p} O_K/p^nO_K$$

induce an isomorphism after taking the inverse limit in the horizontal direction. For this, we note the existence (and uniqueness) of dotted arrows rendering the diagram commutative: this comes from the elementary observation that for $x, y \in O_K$, we have

$$(x \equiv y \pmod{p^{n-1}}) \Rightarrow (x^p \equiv y^p \pmod{p^n}).$$

\[ \square \]

**Corollary 7.** Let $K$ be a completely valued field of residue characteristic $p$. Then we can equip $O_K^\flat$ with the structure of a commutative ring, where the multiplication is defined pointwise and the addition is uniquely determined by the requirement that

$$\{x_n\}_{n \geq 0} + \{y_n\}_{n \geq 0} = \{z_n\}_{n \geq 0} \Rightarrow x_n + y_n \equiv z_n \pmod{p}.$$ 

**Remark 8.** In the situation of Corollary 7, we can describe the addition law on $O_K^\flat$ more explicitly. Suppose we are given elements $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ in $O_K^\flat$. Write $\{x_n\}_{n \geq 0} + \{y_n\}_{n \geq 0} = \{z_n\}_{n \geq 0}$, so that we have $x_m + y_m \equiv z_m \pmod{p}$ for each $n \geq 0$. Writing $z_m = x_m + y_m + pw$ for some $w \in O_K$, we obtain

$$z_0 = (x_m + y_m + pw)^m$$

$$= \sum_{i=0}^{m} \binom{m}{i} (pw)^i (x_m + y_m)^{m-i}$$

$$= (x_m + y_m)^m \pmod{p^m}.$$
It follows that $z_0$ is given concretely as the limit $\lim_{m \to \infty} (x_m + y_m)^p m$. More generally, each $z_n$ is given concretely as $\lim_{m \to \infty} (x_{n+m} + y_{n+m})^{p^n}$.

Note that, to prove Proposition 6, we do not need to assume that $K$ is a perfectoid field: it is enough to assume axioms (A1) and (A2) of Definition 1. However, at this level of generality, the tilt $K^\flat$ might be “too small.”

**Exercise 9.** Let $K = \mathbb{Q}_p$ be the field of $p$-adic rational numbers, equipped with the usual $p$-adic absolute value. Show that $K^\flat = \mathcal{O}_K^\flat$ is isomorphic to $\mathbf{F}_p$.

Our next goal is to show that, when $K$ is a perfectoid field, the tilt $K^\flat$ is very large (Proposition 13).

**Notation 10.** Let $K$ be a completely valued field of residue characteristic $p$ and let $x = \{x_n\}_{n \geq 0}$ be an element of $K^\flat$. We set $x^\sharp = x_0 \in K$. The construction $x \mapsto x^\sharp$ then determines a multiplicative map $\sharp : K^\flat \to K$. For each $x \in K^\flat$, we define $|x|_{K^\flat} = |x^\sharp|_K$.

**Example 11.** Suppose that $K$ is algebraically closed (or, more generally, that every element of $K$ admits a $p$th root). Then the map $x \mapsto x^\sharp$ determines a surjection $K^\flat \to K$.

**Example 12.** Suppose that $K$ is a perfect field of characteristic $p$. Then the map $\sharp : K^\flat \to K$ is bijective.

**Proposition 13.** Let $K$ be a perfectoid field. Then:

1. For every element $x \in \mathcal{O}_K$, there exists an element $x' \in \mathcal{O}_K^\flat$ satisfying $x \equiv x'^\sharp$ (mod $p$).

2. For every element $y \in K$, there exists an element $y' \in K^\flat$ satisfying $|y|_K = |y'|_{K^\flat}$.

**Proof.** Assertion (1) follows from Proposition 6 together with the observation that, if $K$ satisfies axiom (A3), then the transition maps in the diagram

$$\cdots \to \mathcal{O}_K/p \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K/p \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K/p \mathcal{O}_K$$

are surjective.

To prove (2), we may assume without loss of generality we may assume that $y \neq 0$. Using axiom (A4) of Definition 1, we can choose an element $x \in K$ with $|p|_K < |x|_K < 1$. Replacing $x$ by an element which is congruent modulo $p$, we can assume that $x = x'^\sharp$ for some $x' \in K^\flat$ (by virtue of (1)). We are therefore free to modify $y$ by multiplying it by a suitable power of $x$, and can therefore reduce to the case where $|x|_K \leq |y|_K < 1$. In this case, we have $|p|_K < |y|_K < 1$. Using part (1) again, we can choose $y' \in K^\flat$ with $y'^\sharp \equiv y$ (mod $p$), so that $|y|_K = |y'|_{K^\flat}$.

**Exercise 14.** Show that the converse of Proposition 13 is also true: if $K$ is a completely valued field of residue characteristic $p$, then assertion (1) of Proposition 13 implies that $K$ satisfies axiom (A3) of Definition 1, and assertion (2) of Proposition 13 implies that $K$ satisfies axiom (A4) of Definition 1. In other words, the axioms for a perfectoid field are exactly what we need to guarantee that the tilt $K^\flat$ is “sufficiently large.”

Using Proposition 13, we can choose an element $\pi$ in $K^\flat$ such that $0 < |\pi|_{K^\flat} < 1$. For each $n \in \mathbb{Z}$, we have

$$\pi^{-n} \mathcal{O}_K^\flat = \{x \in K^\flat : |x|_{K^\flat} \leq |\pi|_{K^\flat}^{-n}\}$$

It follows that, as a set, we can identify $K^\flat$ with the direct limit

$$\mathcal{O}_K^\flat \xrightarrow{\pi} \mathcal{O}_K^\flat \xrightarrow{\pi} \mathcal{O}_K^\flat \cdots,$$

where the transition maps are given by multiplication by $\pi$. This proves the following:

**Proposition 15.** Let $K$ be a perfectoid field. Then the inclusion $\mathcal{O}_K^\flat \xhookrightarrow{} K^\flat$ extends uniquely to a multiplicative bijection $\mathcal{O}_K^\flat[\pi^{-1}] \simeq K^\flat$. Consequently, there is a unique ring structure on $K^\flat$ which is compatible with its multiplication and which coincides, on $\mathcal{O}_K^\flat$, with the ring structure of Corollary 7.
Exercise 16. Show that the addition law on $K^\flat$ is given in general by the formula

$$\{x_n\}_{n \geq 0} + \{y_n\}_{n \geq 0} = \left\{ \lim_{m \to \infty} (x_{m+n} + y_{m+n})^p^m \right\}_{n \geq 0}$$

Theorem 17. Let $K$ be a perfectoid field. Then $K^\flat$, with the ring structure of Proposition 15 and the map $|\cdot|_{K^\flat} : K^\flat \to \mathbb{R}_{\geq 0}$, is a perfectoid field of characteristic $p$.

Proof. Note that if $\{x_n\}_{n \geq 0}$ is a nonzero element of $K^\flat$, then each $x_n$ is a nonzero element of $K$; it follows that $\{x_n^{-1}\}_{n \geq 0}$ is also an element of $K^\flat$ which is a multiplicative inverse for $\{x_n\}_{n \geq 0}$. This proves that $K^\flat$ is a field. Proposition 6 realizes $O_K^\flat$ as an inverse limit of copies of $O_K/pO_K$ (with transition maps given by the Frobenius). Since $p$ vanishes in $O_K/pO_K$, it vanishes in $O_K^\flat$ and therefore also in $K^\flat$: that is, $K^\flat$ is a field of characteristic $p$. We claim that $|\cdot|_{K^\flat}$ is a non-archimedean absolute value on $K^\flat$. The identities

$$|0|_{K^\flat} = 0 \quad |1|_{K^\flat} = 1 \quad |x \cdot y|_{K^\flat} = |x|_{K^\flat} \cdot |y|_{K^\flat},$$

are immediate from the definition. It will therefore suffice to show that for $x = \{x_n\}_{n \geq 0}$ and $y = \{y_n\}_{n \geq 0} \in K^\flat$, we have

$$|x + y|_{K^\flat} \leq \max(|x|_{K^\flat}, |y|_{K^\flat}).$$

Using the formula of Exercise 16, we are reduced to proving that

$$|(x_m + y_m)^p^m|_K \leq \max(|x_m|_{pK}^m, |y_m|_{pK}^m),$$

which follows (after extracting $p^m$th roots) from the analogous fact for the absolute value $|\cdot|_K$.

The field $K^\flat$ is perfect by construction: every element $(x_0, x_1, x_2, \ldots) \in K^\flat$ has a unique $p$th root, given by the shifted sequence $(x_1, x_2, x_3, \ldots) \in K^\flat$. Moreover, the absolute value on $K^\flat$ is nontrivial because it takes the same values as the absolute value on $K$ (Proposition 13). We will complete the proof by showing that $K^\flat$ is complete. Let us assume that $K$ has characteristic zero (if $K$ has characteristic $p$, then the map $\flat : K^\flat \to K$ is an isomorphism of valued fields and there is nothing to prove). Using Proposition 13, we can choose an element $\pi \in K^\flat$ satisfying $|\pi|_{K^\flat} = |p|_K$. We wish to show that the ring $O_K^\flat$ is $\pi$-adically complete: that is, that it can be realized as the inverse limit of the system

$$\cdots \to O_K^\flat/(\pi^{p^2}) \to O_K^\flat/(\pi^{p^2}) \to O_K^\flat/(\pi^p) \to O_K/(\pi).$$

For each $m \geq 0$, the map of sets

$$O_K^\flat \to \{x = \{x_n\}_{n \geq 0} \mapsto (x_m = (x^{1/p^m})^2)$$

induces a ring homomorphism $O_K^\flat \to O_K/pO_K$ which annihilates $\pi^{p^m}$, and therefore factors through a map $u_m : O_K^\flat/(\pi^{p^m}) \to O_K/pO_K$. These maps fit into a commutative diagram

$$\cdots \to O_K^\flat/(\pi^{p^2}) \to O_K^\flat/(\pi^p) \to O_K^\flat/(\pi) \to O_K/pO_K \xrightarrow{u_2} O_K/pO_K \xrightarrow{u_1} O_K/pO_K \xrightarrow{u_0} O_K/pO_K,$$

where the inverse limit of the lower diagram agrees with $O_K^\flat$ by virtue of Proposition 6. It will therefore suffice to show that each of the maps $u_m$ is an isomorphism. This reduces immediately to the case $m = 0$, where it is a special case of Lemma 18 below.

Lemma 18. Let $K$ be a perfectoid field and let $\pi \in K^\flat$ be a nonzero element satisfying $|p|_K \leq |\pi|_{K^\flat} < 1$. Then the map $\flat : K^\flat \to K$ induces an isomorphism $O_K^\flat/(\pi) \to O_K/(\pi^2)$.

Proof. Surjectivity follows from Proposition 13. To prove injectivity, we note that if $x \in O_K^\flat$ has the property that $x^2 \equiv 0 \pmod{\pi^2}$, then $|x|_{K^\flat} = |x^2|_K \leq |\pi^2|_K = |\pi|_{K^\flat}$, so that $x$ is divisibly by $\pi$ in $O_K^\flat$. \qed