Throughout this lecture, we fix a perfectoid field $C^p$ of characteristic $p$. Our goal in this lecture is to give an “intrinsic” description of $A_{\text{inf}} = W(\mathcal{O}^p_C)$ as a subring of $B$: roughly speaking, it consists of “holomorphic” functions on $Y$ whose value at any point $y \in Y$ belongs to the valuation ring $\mathcal{O}_{K_y}$ of the perfectoid field $K_y$ corresponding to $y$.

**Theorem 1.** Let $f$ be a nonzero element of $B$. The following conditions are equivalent:

1. For each $\rho \in (0, 1)$, we have $|f|_\rho \leq 1$.
2. The element $f$ belongs to the subring $A_{\text{inf}} \subseteq B$.

**Corollary 2.** Let $f$ be a nonzero element of $B$. Then:

- The element $f$ belongs to the localization $A_{\text{inf}}\left[\frac{1}{p}\right]$ if and only if there exists an integer $n$ such that $|f|_\rho \leq \rho^n$ for all $\rho \in (0, 1)$.
- The element $f$ belongs to the localization $A_{\text{inf}}\left[\frac{1}{p^2}, \frac{1}{p}\right]$ if and only if there exists a constant $C > 0$ satisfying $|f|_\rho \leq C \rho^n$ for all $\rho \in (0, 1)$.
- The element $f$ belongs to the localization $A_{\text{inf}}\left[\frac{1}{p^3}, \frac{1}{p^2}, \frac{1}{p}\right]$ if and only if there exists a constant $C > 0$ and an integer $n$ satisfying $|f|_\rho \leq C \rho^n$ for all $\rho \in (0, 1)$.

We will deduce Theorem 1 from the following weaker assertion:

**Lemma 3.** Let $f$ be an element of $B$. Suppose that there exists an integer $m$ such that $|f|_\rho \leq \rho^m$ for all $0 < \rho < 1$. Then we can write $f = [c]p^m + g$, where $c \in \mathcal{O}^p_C$ and $g$ satisfies an inequality of the form $|g|_\rho \leq \rho^{m+1}$.

**Proof of Theorem 1 from Lemma 3.** The implication $(2) \Rightarrow (1)$ is immediate. Conversely, suppose that $(1)$ is satisfied, and set $f_0 = f$. Applying Lemma 3, we can write $f_0 = [c_0] + f_1$, where $[c_0] \in \mathcal{O}^p_C$, and $f_1$ satisfies $|f_1|_\rho \leq \rho$ for all $\rho \in (0, 1)$. Applying Lemma 3 again, we can write $f_1 = [c_1]p + f_2$, where $[c_1] \in \mathcal{O}^p_C$, and $f_2$ satisfies $|f_2|_\rho \leq \rho$ for all $\rho \in (0, 1)$. Continuing in this way, we obtain a sequence of elements $f_0, f_1, f_2, \ldots \in B$ and $c_0, c_1, c_2, \ldots \in \mathcal{O}^p_C$, satisfying

$$f_0 = [c_0] + [c_1]p + \cdots + [c_{n-1}]p^{n-1} + f_n$$

with $|f_n|_\rho \leq \rho^n$.

Note that the sequence $\{f_n\}_{n \geq 0}$ converges to zero with respect to the each of the Gauss norms $| \bullet |_\rho$. It follows that the infinite sum $\sum_{n \geq 0}[c_n]p^n$ converges in $B$ to $f$, so that $f$ belongs to $A_{\text{inf}}$ as desired. \qed
Proof of Lemma 3. Replacing \( f \) by \( \frac{f}{\rho^n} \), we can reduce to the case \( m = 0 \). In this case, we have an element \( f \in B \) satisfying \( |f|_\rho \leq 1 \) for all \( \rho \in (0,1) \); we wish to write \( f = [c] + g \) for \( c \in \mathcal{O}_C^\flat \), where \( g \) satisfies \( |g|_\rho \leq \rho \) for all \( \rho \in (0,1) \).

Choose a sequence \( f_1, f_2, \ldots \in A_{\text{inf}}[\frac{1}{p}, \frac{1}{|\rho|}] \) which converges to \( f \) in \( B \). Each \( f_i \) then admits a unique Teichmüller expansion
\[
f_i = \sum_{n \geq -\infty} [c_{n,i}] p^n.
\]
Set \( f_i^+ = \sum_{n \geq 0} [c_{n,i}] p^n \). We claim that the sequence \( f_1^+, f_2^+, \ldots \) also converges to \( f \) in \( B \). To prove this, we must show that for each \( \rho \in (0,1) \), we have
\[
\lim_{i \to \infty} |f_i - f_i^+|_\rho = 0.
\]

Let \( \epsilon \) be a small positive real number. Then the sequence \( f_1, f_2, \ldots \) converges to \( f \) with respect to the Gauss norm \(| \cdot |_{\epsilon,\rho} \). It follows that, for \( i \) sufficiently large (depending on \( \epsilon \)), we have
\[
|f_i|_{\epsilon,\rho} = |f|_{\epsilon,\rho} \leq 1.
\]
For such \( i \), we have
\[
|c_{-n,i}|_{C,\rho} (\epsilon \rho)^{-n} \leq 1.
\]
If \( n \) is positive, this gives
\[
|c_{-n,i}|_{C,\rho} \rho^{-n} \leq \epsilon^n \leq \epsilon.
\]
We therefore have
\[
|f_i - f_i^+|_\rho = \sup_{n > 0} (|c_{-n,i}|_{C,\rho} \rho^{-n}) \leq \epsilon
\]
for sufficiently large \( i \).

Replacing the sequence \( \{f_i\} \) with \( \{f_i^+\} \), we may assume that each \( f_i \) admits a Teichmüller expansion of the form
\[
f_i = \sum_{n \geq 0} [c_{n,i}] p^n.
\]
Then, for every pair of indices \( i \) and \( j \), the difference \( f_i - f_j \) admits a Teichmüller expansion of the form \( [c_{0,i} - c_{0,j}] + \text{higher order terms} \). For any \( \rho \in (0,1) \), we have
\[
|f_i - f_j|_\rho \geq |c_{0,i} - c_{0,j}|_{C,\rho}.
\]
Since the sequence \( \{f_i\} \) is Cauchy with respect to the Gauss norm \(| \cdot |_{\rho} \), it follows that \( \{c_{0,i}\} \) is a Cauchy sequence in the field \( C^\flat \). Since \( C^\flat \) is complete, this Cauchy sequence converges to some element \( c \in C^\flat \). Moreover, for \( i \gg 0 \), we have
\[
|c_{0,i}|_{C^\flat} \leq |f_i|_\rho = |f|_\rho \leq 1,
\]
so that \( c_{0,i} \) belongs to \( \mathcal{O}_C^\flat \) (for \( i \gg 0 \)) and therefore \( c \in \mathcal{O}_C^\flat \).

Exercise 4. Show that, if \( \{c_i\} \) is a Cauchy sequence in \( \mathcal{O}_C^\flat \), converging to a point \( c \in \mathcal{O}_C^\flat \), then we have
\[
[c] = \lim_{i \to \infty} [c_i] \quad \text{in the ring } B.
\]
For each \( i \), set \( g_i = f_i - [c_{0,i}] = \sum_{n > 0} [c_{n,i}] p^n \). Applying the exercise, we see that the limit \( \lim_{i \to \infty} g_i \) exists and is given by
\[
\lim_{i \to \infty} g_i = (\lim_{i \to \infty} f_i) - (\lim_{i \to \infty} [c_{0,i}]) = f - [c].
\]
That is, we can write \( f = [c] + g \), where \( g = \lim_{i \to \infty} g_i \). We will complete the proof by showing that \( |g|_\rho \leq \rho \) for all \( \rho \in (0,1) \), or equivalently that \( v_s(g) \geq s \) for all \( s \in \mathbb{R}_{>0} \).
Let us assume that \( g \neq 0 \) (otherwise there is nothing to prove). Passing to a subsequence, we may then also assume that \( g_i \neq 0 \) for all \( i \). Each \( g_i \) admits a Teichmüller expansion where only positive powers of \( p \) occur, so that the piecewise linear function \( v_s(g_i) \) has strictly positive slopes. When restricted to any compact interval \( I \subseteq \mathbb{R}_{>0} \), the function \( v_s(g) \) agrees with \( v_s(g_i) \) for \( i \gg 0 \). It follows that the piecewise linear function \( s \mapsto v_s(g) \) also has strictly positive (and integral) slopes. Suppose, for a contradiction, that there exists some \( s > 0 \) such that \( v_s(g) < s \). Choose \( 0 < s' < s \) such that \( v_s(g) - s + s' < 0 \). Since the function \( v_s(g) \) is piecewise linear with slopes \( \geq 1 \) everywhere, we have

\[
v_{s'}(g) \leq v_s(g) - s + s' < 0.
\]

Setting \( \rho' = e^{-s'} \), we have \( |g|_{\rho'} > 1 \). Then

\[
1 < |g|_{\rho'} = |f - [c]|_{\rho'} \leq \max(|f|_{\rho'}, |[c]|_{\rho'}) = \max(|f|_{\rho'}, |c|_{C'}) \leq 1,
\]

which is a contradiction. \( \square \)

From Theorem 1, it is easy to describe the invariant subring \( B^{\varphi=1} \subseteq B \):

**Theorem 5.** The unit map \( Q_p \to B^{\varphi=1} \) is an isomorphism.

**Lemma 6.** Let \( f \) be a nonzero element of \( B^{\varphi=1} \). Then there exists an integer \( n \) such that \( |f|_\rho = \rho^n \) for all \( 0 < \rho < 1 \).

**Proof.** Note that for \( 0 < \rho < 1 \), we have

\[
|f|^p_\rho = |\varphi(f)|_{\rho^p} = |f|_{\rho^p}.
\]

In other words, the function \( s \mapsto v_s(f) \) satisfies the identity \( v_{ps}(f) = pv_s(f) \). Differentiating both sides (on the left) with respect to \( s \) and dividing by \( p \), we obtain \( \partial_{-} v_{ps}(f) = \partial_{-} v_s(f) \). Since the function \( s \mapsto v_s(f) \) is concave, the function \( s \mapsto \partial_{-} v_s(f) \) is nondecreasing; the above equality implies that it is constant. In other words, \( s \mapsto v_s(f) \) is a linear function of \( s \), which we can write as \( v_s(f) = ns + r \) for some integer \( n \) and some real number \( r \). The equality \( v_{ps}(f) = pv_s(f) \) then implies that \( r = 0 \), so that \( v_s(f) = ns \) for all \( s > 0 \) and therefore \( |f|_\rho = \rho^n \) for all \( 0 < \rho < 1 \). \( \square \)

**Proof of Theorem 5.** Let \( f \) be a nonzero element of \( B^{\varphi=1} \). It follows from Lemma 6 and Corollary 2 that \( f \) belongs to the subring \( \mathcal{A}_{\inf[\frac{1}{2}]} \subseteq B \). That is, \( f \) admits a unique Teichmüller expansion

\[
f = \sum_{n \gg -\infty} [c_n]p^n,
\]

where each \( c_n \) belongs to \( \mathcal{O}_{C'}^p \). We then have

\[
\sum_{n \gg -\infty} [c_n]p^n = f = \varphi(f) = \sum_{n \gg -\infty} [c^n_p]p^n,
\]

so that each coefficient \( c_n \) satisfies \( c_n = c^n_p \) in the field \( C'^p \), and therefore belongs to the finite field \( \mathbb{F}_p \subseteq C'^p \).

The equality \( f = \sum_{n \gg -\infty} [c_n]p^n \) now shows that \( f \) belongs to \( Q_p = W(\mathbb{F}_p)[\frac{1}{p}] \), as desired. \( \square \)