Math 155 (Lecture 6)

September 13, 2011

In this lecture, we will continue our discussion of natural operations on species.

**Definition 1.** Let $S$ and $T$ be species. We define a new species $ST$, called the *product* of $S$ and $T$, as follows:

(a) Let $I$ be a finite set. We define $(ST)[I]$ to be the finite set given by the disjoint union

$$
\prod_{I = I_0 \cup I_1} S[I_0] \times T[I_1].
$$

Here the product is taken over all decompositions of $I$ into disjoint subsets $I_0$ and $I_1$.

(b) If $\pi : I \to J$ is a bijection of finite sets, we take $(ST)[\pi] : (ST)[I] \to (ST)[J]$ to be the bijection given by the disjoint union of the maps

$S[I_0] \times T[I_1] \to S[\pi(I_0)] \times T[\pi(I_1)]$

determined by the bijections $I_0 \to \pi(I_0)$ and $I_1 \to \pi(I_1)$.

**Example 2.** Suppose we are given a finite set $C$ of colors. For each finite set $I$, let $S^C[I]$ denote the set of all colorings of $I$ by the set $C$: that is, the set $C^I$ of all maps $I \to C$. Then $S^C$ is a species, called the *species of coloring by $C$*.

Now suppose that $C$ is given as a disjoint union of subsets $C_0$ and $C_1$. Then every coloring $f : I \to C$ of a set $I$ determines a decomposition of $I$ into disjoint subsets $I_0$ and $I_1$, given by

$$I_0 = f^{-1}C_0 \quad I_1 = f^{-1}C_1.$$ 

Moreover, to recover the coloring of $I$, we need this decomposition together with a coloring of $I_0$ by $C_0$ and a coloring of $I_1$ by $C_1$. In other words, we have a bijection

$$S^C[I] = \prod_{I = I_0 \cup I_1} S^{C_0}[I_0] \times S^{C_1}[I_1].$$

These bijections give an *isomorphism* between the species $S^C$ and the product species $S^{C_0}S^{C_1}$.

**Proposition 3.** Let $S$ and $T$ be species with exponential generating functions $F_S(x)$ and $F_T(x)$. Then the product species $ST$ has exponential generating function $F_{ST}(x) = F_S(x)F_T(x)$.

**Proof.** We compute

$$F_S(x)F_T(x) = \left(\sum_{p \geq 0} \frac{|S[p]|}{p!} x^p\right) \left(\sum_{q \geq 0} \frac{|T[q]|}{q!} x^q\right) = \sum_{p,q \geq 0} \frac{|S[p]|}{p!} \frac{|T[q]|}{q!} x^{p+q} = \sum_{n \geq 0} \sum_{p+q=n} \frac{n!}{p!q!} \frac{|S[p]|}{p!} \frac{|T[q]|}{q!} x^n.$$
On the other hand, we have

\[
F_{ST}(x) = \sum_{n \geq 0} \frac{|(ST)[(n)]|}{n!} x^n = \sum_{n \geq 0} \sum_{(n) = I_0 \cup I_1} \frac{|S[I_0] \times T[I_1]|}{n!} x^n.
\]

Every decomposition \( (n) = I_0 \cup I_1 \) into disjoint subsets determines a pair of natural numbers \( p = |I_0|, q = |I_1| \) with \( p + q = n \), and we have \( |S[I_0] \times T[I_1]| = |S[(p)] \times T[(q)]| \). Each of these factors occurs precisely \( \binom{n}{p} = \frac{n!}{p!q!} = \binom{n}{q} \) times in the second sum, so that

\[
F_{ST}(x) = \sum_{n \geq 0} \sum_{p+q=n} \frac{n!}{p!q!} \frac{|S[(p)] \times T[(q)]|}{n!} x^n = F_S(x)F_T(x)
\]
as desired.

**Example 4.** Let \( C \) be a finite set with \( c \) elements, and let \( S^C \) be the species of \( C \)-colorings appearing in Example 2. Then \( |S^C[I]| = |C^I| = c^{|I|} \) for every finite set \( I \), so that

\[
F_{S^C}(x) = \sum_{n \geq 0} \frac{c^n}{n!} x^n = e^{cx}.
\]

If \( C \) is given as a disjoint union of subsets \( C_0 \) and \( C_1 \) having sizes \( c_0 \) and \( c_1 \), then we have an isomorphism of species \( S^C \simeq S^{C_0}S^{C_1} \). Proposition 3 gives an identity of generating functions

\[
F_{SC}(x) = F_{S^C_0}(x)F_{S^C_1}(x),
\]
which reduces to the familiar identity

\[
e^{cx} = e^{c_0x}e^{c_1x}.
\]

**Example 5.** Let \( S \) be the species of undecorated sets: \( S[I] \) has a single element for every finite set \( I \). The exponential generating function \( F_S \) is given by \( \sum_{n \geq 0} \frac{1}{n!} x^n = e^x \). Let \( T \) be the species of derangements. The product \( ST \) assigns to each finite set \( I \) the collection all decompositions \( I = I_0 \cup I_1 \), together with a derangement of \( I_1 \). It follows that \( ST \) is isomorphic to the species of permutations (every permutation of the set \( I \) has a fixed point set \( I_0 \subseteq I \), and determines a derangement of the complement \( I_1 = I - I_0 \)). It follows that

\[
\frac{1}{1-x} = F_{ST}(x) = F_S(x)F_T(x) = e^xF_T(x).
\]

We therefore recover our formula

\[
F_T(x) = \frac{e^{-x}}{1-x}
\]
for the exponential generating function of derangements.

**Definition 6.** Let \( S \) and \( T \) be species, and assume that \( T[\emptyset] = \emptyset \). We define a new species \( S \circ T \) as follows. For each finite set \( I \), let \( (S \circ T)[I] \) denote the set of all triples \((\sim, x, \{y_J\}_{J \in I/\sim})\) where \( \sim \) is an equivalence relation on \( I \), \( x \in S[I/\sim] \), and for each \( J \in I/\sim \), \( y_J \) is an element of \( T[J] \) (here we identify the elements of \( I/\sim \) with equivalence classes in \( I \)).

Our goal in this section is to prove the following result:

**Theorem 7.** Let \( S \) and \( T \) be species, and assume that \( T[\emptyset] = \emptyset \). Then we have an equality of power series \( F_{S \circ T}(x) = F_S(F_T(x)) \). In other words, the construction \( S \mapsto F_S \) is compatible with composition.
Example 8. Let $S$ be the species of nonempty sets: that is, we have

$$S[I] = \begin{cases} \{\ast\} & \text{if } I \neq \emptyset \\ \emptyset & \text{if } I = \emptyset. \end{cases}$$

For every finite set $I$, we see that $(S \circ S)[I]$ is the set of all equivalence relations $\sim$ on $I$ such that each equivalence class is nonempty (this condition is automatic) and $I/\sim$ is nonempty (which is true if and only if $I$ is nonempty). We therefore have

$$(S \circ S)[I] \simeq \begin{cases} \emptyset & \text{if } I = \emptyset \\ \text{partitions of } I & \text{if } I \neq \emptyset. \end{cases}$$

The exponential generating function for $S$ is given by $FS = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1$. It follows from Theorem 7 that the exponential generating function for $S \circ S$ is given by

$$FS\circ S(x) = FS(FS(x)) = FS(e^x - 1) = e^{e^x-1} - 1.$$

From this, we recover our formula for the exponential generating function for the Bell numbers:

$$\sum_{n \geq 0} \frac{b_n}{n!} x^n = 1 + FS\circ S(x) = 1 + e^{e^x-1} - 1 = e^{e^x-1}.$$

Warning 9. To make sense of Theorem 7, we need to define the composition $FS(F_T(x))$. This is given by

$$\sum_{n \geq 0} \frac{S[(n)]}{n!} F_T(x)^n.$$

This sum is sensible because of our assumption that $T[\emptyset] = \emptyset$, which guarantees that the constant term of $F_T(x)$ vanishes (so that $F_T(x)^n$ is divisible by $x^n$).

Proof of Theorem 7. For each natural number $k \geq 0$, let $X_k$ denote the species which assigns to each finite set $I$ the subset $X_k[I] \subseteq (S \circ T)[I]$ consisting of those triples $(\sim, x, \{y_j\})$ where the quotient $I/\sim$ has exactly $k$ elements. Then $(S \circ T)[I]$ is a disjoint union of the subsets $X_k[I]$ as $k$ ranges over all natural numbers (note that the set $X_k[I]$ is empty if $k > |I|$), so that

$$FS\circ T(x) = \sum_{k \geq 0} F_{X_k}(x).$$

Similarly, we have

$$FS(F_T(x)) = \sum_{k \geq 0} \frac{S[(k)]}{k!} F_T(x)^k.$$

We will complete the proof by showing that

$$F_{X_k}(x) = \frac{S[(k)]}{k!} F_T(x)^k.$$

Let $X_k[I]$ denote the species which assigns to each finite set $I$ the disjoint union

$$\prod_{I = I_1 \cup \cdots \cup I_k} T[I_1] \times \cdots \times T[I_k] \times S[(k)].$$

The sets $X_k[I]$ and $X_k[I]$ are almost the same: the only difference is that an element of $X_k[I]$ specifies a partition of $I$ into unlabelled pieces, while $X_k[I]$ specifies a partition into labelled pieces. We therefore have

$$|X_k[I]| = k!|X_k[I]|,$$
so that $F_{\Xi_k}(x) = k! F_{X_k}(x)$.

Using Proposition 3 repeatedly, we see that $F_T(x)^k$ is the exponential generating function for the species $T^k$. Unwinding the definitions, we see that $T^k$ assigns to a finite set $I$ the disjoint union

$$\coprod_{I = I_1 \cup \cdots \cup I_k} T[I_1] \times \cdots \times T[I_k],$$

where the coproduct is taken over all decompositions of $I$ into disjoint labelled subsets $I_1, \ldots, I_k$. We therefore have

$$|\Xi_k[I]| = |T^k[I]| |S[\langle k \rangle]|$$

for each finite set $I$. Passing to generating functions, we get

$$F_T(x)^k |S[\langle k \rangle]| = F_{\Xi_k}(x) = k! F_{X_k}(x),$$

so that $F_{X_k}(x) = \frac{S[\langle k \rangle]}{k!} F_T(x)^k$ as desired. \hfill $\square$