If $G$ is a graph and $S$ is a set of vertices of $G$, we let $G - S$ denote the graph obtained from $G$ by removing the vertex set $S$. Our first goal in this lecture is to prove the following:

**Theorem 1** (Tutte). Let $G$ be a finite graph with vertex set $V$. Then $G$ has a perfect matching if and only if the following condition is satisfied, for every subset $S \subseteq V$:

1. The number of connected components of $G - S$ having an odd number of vertices is $\leq |S|$.

We saw in the last lecture that (1) is necessary. Note also that (1) implies that $G$ has an even number of vertices (take $S = \emptyset$).

We now prove the sufficiency. Assume that $G$ satisfies (1); we wish to show that $G$ has a perfect matching. Let us fix the number of vertices of $G$, and work by reverse induction on the number of edges of $G$. That is, we will assume that Theorem 1 is valid for any graph $G'$ having the same number of vertices as $G$ but more edges than $G$. In particular, if $x$ and $y$ are vertices of $G$ which are not connected by an edge, and $G'$ is the graph obtained from $G$ by adjoining an edge joining $x$ to $y$, then we may assume that Theorem 1 is valid for the graph $G'$. Note that if $G$ satisfies (1), then $G'$ also satisfies (1): if $S$ is any set of vertices of $G'$, then either $G' - S \simeq G - S$ or $G' - S$ is obtained by adding an edge to $G - S$. In either case, the number of odd components of $G' - S$ must be smaller than the number of odd components of $G - S$.

We now take $S$ to be the set of vertices of $G$ which are connected to every other vertex of $G$. There are two cases to consider:

(a) Every connected component of $G - S$ is a complete graph. We can then construct a perfect matching for $G$ as follows:

1. Choose a perfect matching for each connected component of $G - S$ having even size.
2. For each connected component $H$ of $G - S$ having odd size, choose a vertex $v_H \in H$ and a perfect matching for $H - \{v_H\}$. Choose also a vertex $w_H \in S$, and add to our matching the edge joining $v_H$ to $w_H$. Since $G$ satisfies (1), we can assume that the vertices $w_H$ are all distinct.
3. Since $G$ has even size, there are an even number of remaining vertices, each of which belongs to $S$. These vertices are all adjacent to one another, so we can complete our matching by arbitrarily dividing them into pairs.

(b) Suppose that some connected component of $G - S$ is not a complete graph: that is, the relation of adjacency is not an equivalence relation on the vertices of $G - S$. Then transitivity must fail: that is, we can find edges $x, y,$ and $z$ of $G - S$ such that $x$ is adjacent to $y$, $y$ is adjacent to $z$, but $x$ is not adjacent to $z$. Since $y \notin S$, we can also choose a vertex $w$ such that $y$ is not adjacent to $w$.

Let $G'$ be the graph obtained from $G$ by adding an edge from $w$ to $y$, and $G''$ the graph obtained from $G$ by adding an edge from $x$ to $z$. Then $G'$ and $G''$ both have more vertices than $G$. Applying the inductive hypothesis, we deduce that $G'$ and $G''$ have perfect matchings $M'$ and $M''$. We may assume that $M'$ contains the edge $\{w, y\}$; otherwise, it is a perfect matching for the graph $G$. Similarly, we may assume that $M''$ contains the edge $\{x, z\}$. 

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Let us now consider the graph $H$ with the set of edges $M' \cup M''$. Note that every vertex in this graph belongs to either a single edge (if it belongs to an edge of $M' \cap M''$), or to two edges (one of which belongs to $M'$ and one to $M''$). It follows that $H$ can be written as a disjoint union of edges belonging to $M' \cap M''$ and cycles of even length, consisting of edges which belong alternatively to $M$ and to $M'$. Note that the edges $\{w, y\}$ and $\{x, z\}$ cannot belong to $M' \cap M''$, so they are contained in cycles of $C, D \subseteq H$. Let $C_1, C_2, \ldots, C_m$ be the remaining components of $H$. Then $C_1, C_2, \ldots, C_m$ are contained in $G$, and each admits a perfect matching. Let $G_0$ be the graph obtained from $G$ by removing the vertices of $C_1, C_2, \ldots, C_m$. It will now suffice to show that $G_0$ admits a perfect matching.

Suppose first that $C \neq D$. Let $C_0$ be the graph obtained from $C$ by removing the vertex $\{w, y\}$, and define $D_0 \subseteq D$ similarly. Then $C_0$ and $D_0$ are chains of even length, and therefore admit perfect matchings. The union of these perfect matchings is then a perfect matching for $G_0$.

Let us now suppose that $C = D$. Then $C$ is a cycle of even length containing $w, y, x,$ and $z$, with $w$ adjacent to $y$ and $x$ adjacent to $z$. Without loss of generality, we may assume that this cycle has the form

$$v_0 = y, v_1 = w, v_2, v_3, \ldots, v_m = x, v_{m+1} = z, v_{m+2}, \ldots, v_n = y$$

for some even number $n$. Here the edge $\{v_i, v_{i+1}\}$ belongs to $M'$ if $i$ is even, and to $M''$ if $i$ is odd. In particular, $m$ is an odd number. In this case, we have a perfect matching for $G_0$ given by the edges

$$\{y, v_2\}, \{v_3, v_4\}, \ldots, \{v_{m-2}, v_{m-1}\}, \{x, y\}, \{z, v_{m+2}\}, \{v_{m+3}, v_{m+4}\}, \ldots, \{v_n, v_{n-1}\}.$$

This completes the proof of Theorem 1.

**Definition 2.** Let $G$ be a graph. An Eulerian cycle in $G$ is a cycle (possibly self-intersecting)

$$v_0, v_1, \ldots, v_n = v_0$$

which uses each edge of $G$ exactly once: that is, each edge of $G$ has the form $\{v_i, v_{i+1}\}$ for a unique value of $i$.

**Question 3.** When does a graph $G$ admit an Eulerian cycle?

There are two obvious constraints:

(1) If $G$ is a graph with an Eulerian cycle, then there must exist a connected component of $G$ which contains all the edges of $G$ (that is, $G$ is the union of a connected graph with a collection of isolated vertices).

(2) If $v_0, v_1, \ldots, v_n = v_0$ is an Eulerian cycle in $G$, then each vertex of $G$ must have even degree. In fact, the degree of a vertex $w$ is twice the number of occurrences of $w$ in the list $v_0, v_1, \ldots, v_{n-1}$.

**Theorem 4.** If $G$ is a finite graph satisfying conditions (1) and (2), then $G$ admits an Eulerian cycle.

Theorem 4 is a special case of a more general result about Eulerian paths. An Eulerian path in $G$ is a path $v_0, v_1, \ldots, v_n$ which uses each edge exactly once. However, we do not assume that $v_0 = v_n$. Any graph $G$ which admits an Eulerian path must satisfy condition (1), together with the following slightly weaker version of condition (2):

(2') The graph $G$ has at most two vertices of odd degree. In fact, if $v_0, \ldots, v_n$ is an Eulerian path in $G$, then every vertex of $G$ other than $v_0$ and $v_n$ must have even degree. Moreover, either $v_0 \neq v_n$ and both have odd degree, or $v_0 = v_n$ has even degree (in which case we have an Eulerian cycle).

Theorem 4 is a consequence of the following:

**Theorem 5.** If $G$ is a finite graph satisfying conditions (1) and (2'), then $G$ admits an Eulerian path.
Remark 6. Let $G$ be any finite graph. For each vertex $v$ of $G$, let $d(v)$ denote the degree of $v$. Then $\sum_v d(v)$ is twice the number of edges of $G$ (since each edge is counted twice). In particular, $\sum_v d(v)$ is an even number. It follows that $G$ must have an even number of vertices of odd degree. In particular, if (2') is satisfied, then either every vertex of $G$ has even degree, or $G$ has exactly two vertices of odd degree.

Proof of Theorem 5. We may assume without loss of generality that $G$ is connected. If $G$ has two vertices of odd degree, denote them by $v$ and $w$. Otherwise, choose any vertex $v \in G$ and set $w = v$. We will show that there is an Eulerian path in $G$ starting and ending at $w$. The proof proceeds by induction on the number of edges of $G$.

Suppose first that there exists an edge $\{v, v'\}$ of $G$ with the following property: the graph $H$ obtained by removing the edge $\{v, v'\}$ is connected. In this case, the inductive hypothesis implies that there is an Eulerian path from $v'$ to $w$ in the graph $H$. Appending the edge $\{v, v'\}$ to the beginning of this path, we obtain an Eulerian path from $v$ to $w$ in the graph $G$.

We may therefore assume that $d \geq 2$. The graph $G - \{v\}$ is a union of connected components $G_1, G_2, \ldots, G_d$. Each of these graphs has an even number of vertices having odd degree. Moreover, each $G_i$ has exactly one vertex which is connected to $v$. It follows that each $G_i$ has an odd number of vertices which have odd degree in the graph $G$. In particular, each $G_i$ has at least one vertex of odd degree in $v$. Since no vertex of $G$ other than $v$ and $w$ can have odd degree, we conclude that $d \leq 1$. That is, there is a unique edge $\{v, v'\}$ containing $v$. Then $G - \{v\}$ is connected, and the inductive hypothesis implies that there is an Eulerian path from $v'$ to $w$ in $G - \{v\}$. Appending $v$ to the beginning of this path, we obtain an Eulerian path from $v$ to $w$ in $G$. \qed