Definition 1. Let $G$ be a graph with vertex set $V$. We say that $G$ is bipartite if there exists a decomposition $V = V_0 \cup V_1$ into disjoint subsets, such that every edge consists of a vertex from $V_0$ together with a vertex from $V_1$. (That is, neither $V_0$ nor $V_1$ contains a pair of adjacent vertices.)

Remark 2. A disjoint union of bipartite graphs is bipartite.

Proposition 3. Let $G$ be a graph. The following conditions are equivalent:

1. The graph $G$ is bipartite.
2. The graph $G$ contains no cycles of odd length.

Proof. Suppose first that $G$ is bipartite, and let $v_0, v_1, \ldots, v_n = v_0$ be a cycle of $G$. Since $G$ is bipartite, its vertex set $V$ can be partitioned into subsets $V_0$ and $V_1$ as in Definition 1. We may assume without loss of generality that $v_0 \in V_0$. Then $v_1$ is adjacent to $v_0$, so that $v_1 \in V_1$. The same argument shows that $v_2 \in V_0$, $v_3 \in V_1$, and so forth. Since $v_n = v_0 \in V_0$, we conclude that $n$ is even.

Now suppose that condition (2) is satisfied. We wish to show that $G$ is bipartite. Since the collection of bipartite graphs is closed under disjoint unions (Remark 2), we may suppose also that $G$ is connected. Fix a vertex $v \in V$. For each $w \in V$, let $d(v, w)$ denote the distance from $v$ to $w$: that is, the length of the shortest path from $v$ to $w$. Let $V_0 = \{w \in V : d(v, w) \text{ is even} \}$ and $V_1 = \{w \in V : d(v, w) \text{ is odd} \}$. We claim that the decomposition $V = V_0 \cup V_1$ satisfies the requirements of Definition 1. Suppose otherwise: then there exists a pair of adjacent vertices $w, w' \in V$ such that either $w, w' \in V_0$ or $w, w' \in V_1$. Let us assume that $w, w' \in V_0$ (the other case can be handled in a similar way). Then there exists a path $v = v_0, v_1, \ldots, v_m = w$ of even length. Similarly, there exists a path $v = v'_0, v'_1, \ldots, v'_n = w'$ of even length. Then the cycle $v = v_0, v_1, \ldots, v_m = w, w' = v'_n, v'_{n-1}, \ldots, v'_0 = v$ has length $m + n + 1$ which is an odd number, contradicting assumption (2).

Now suppose that $G$ is a bipartite graph with vertex set $V$, and that we are given a decomposition $V = V_0 \cup V_1$ satisfying the requirements of Definition 1. A matching of $V_0$ to $V_1$ is an injective map $f : V_0 \rightarrow V_1$ such that $f(v)$ is adjacent to $v$, for each $v \in V_0$.

Question 4 (Marriage Problem). Given a bipartite graph $G$ as above, when does there exists a matching $f : V_0 \rightarrow V_1$?

Remark 5. The terminology of Question 4 is motivated as follows: we imagine that $V_0$ is the set of men in some village and $V_1$ the set of women in some village, and that a pair of vertices $v \in V_0$, $w \in V_1$ are adjacent if they are willing to marry. Then Question 4 asks if some matchmaker could arrange a marriage for every man in the village, with no two men marrying the same woman.

Remark 6. As formulated in Question 4, the marriage problem is not symmetric. However, if $V_0$ and $V_1$ have the same size, then any injection from $V_0$ to $V_1$ is a bijection, whose inverse is an injection from $V_1$ to $V_0$. Thus, in this special case, the problem is symmetric.

There are some situations which one can obviously not solve the marriage problem of Question 4:
Example 7. If $|V_0| > |V_1|$, then there cannot exist any map $f : V_0 \to V_1$. It follows that there cannot be a matching between $V_0$ and $V_1$.

Example 8. If there is some vertex $v \in V_0$ which is not adjacent to any vertex in $V_1$, then there cannot be a matching from $V_0$ to $V_1$.

We can simultaneously rule out the bad situations described in Examples 7 and 8 with the following assumption:

(*) For every subset $S \subseteq V_0$, let $S^+ \subseteq V_1$ be the set $\{v \in V_1 : v$ is adjacent to some $w \in S\}$. Then $|S^+| \geq |S|$.

When $S$ has a single element, this says that every vertex of $V_0$ is adjacent to some vertex of $V_1$. When $S = V_0$, it guarantees that $|V_1| \geq |V_0|$.

Theorem 9 (Hall’s Marriage Theorem). Let $G$ be a bipartite graph with vertex set $V = V_0 \cup V_1$ as above. Then there is a matching $f : V_0 \to V_1$ if and only if condition (*) is satisfied.

Proof. We first prove the “only if” direction. Suppose there is a matching $f : V_0 \to V_1$, and let $S \subseteq V_0$. Then $f(S) \subseteq S^+$, so that $|S^+| \geq |f(S)| = |S|$.

The hard part is to prove the “if” direction. We will prove, using induction on the integer $|V_0|$, that condition (*) implies the existence of a matching $V_0 \to V_1$. We consider two cases:

(a) Suppose that there exists a nonempty proper subset $S \subseteq V_0$ such that $|S^+| = |S|$. Applying the inductive hypothesis, we can find a matching $f : S \to S^+$. Let $G'$ be the graph obtained from $G$ by removing $S$ and $f(S)$, so that the set of vertices of $G'$ can be decomposed into subsets $W_0 = V_0 - S$ and $W_1 = V_1 - f(S)$. For each subset $T \subseteq W_0$, let $T^+ \subseteq W_1$ be defined as in (*). Then $(S \cup T)^+ \subseteq T^+ \cup S^+ = T^+ \cup f(S)$, so that $|T^+| = (|S \cup T|^+) - |f(S)| \geq |S \cup T| - |S| = |T|$. It follows that the graph $G'$ satisfies condition (*), so that the inductive hypothesis guarantees the existence of a matching $g : W_0 \to W_1$. Together, the maps $f$ and $g$ determine a matching $V_0 \to V_1$.

(b) Suppose that for every nonempty proper subset $S \subseteq V_0$, we have $|S^+| > |S|$. If $V_0$ is empty, there is nothing to prove. Otherwise, we can choose a vertex $v \in V_0$. Since $\{v\}^+$ is nonempty, we can choose a vertex $w \in V_1$ adjacent to $v$. Let $G'$ be the graph obtained from $G$ by removing the vertices $v$ and $w$, so that the vertices of $G'$ can be decomposed into subsets $W_0 = V_0 - \{v\}$ and $W_1 = V_1 - \{w\}$. For each nonempty subset $S \subseteq W_1$, the set $S^+ = \{u \in W_1 : u$ is adjacent to some $t \in T\}$ coincides with $S^+ - \{w\}$, where $S^+$ is computed in the graph $G$. Since $S$ is a proper subset of $V_0$, we have $|S^+| \geq |S^+| - 1 > |S| - 1$, so that $|S^+| \geq |S|$. Thus the graph $G'$ satisfies (*), so the inductive hypothesis gives us a matching $g : W_0 \to W_1$. This extends to a matching $f : V_0 \to V_1$ by setting $f(v) = w$.

Here is a reformulation of the marriage theorem which does not mention graphs:

Theorem 10. Let $X$ be a finite set, and suppose we are given subsets $Y_1, Y_2, \ldots, Y_m \subseteq X$. Assume that:

(*) For every subset $S \subseteq \{1, \ldots, m\}$, the set $\bigcup_{i \in S} Y_i$ has cardinality at least $|S|$.

Then there exists a sequence of elements $y_1 \in Y_1$, $y_2 \in Y_2$, \ldots, with $y_i \neq y_j$ for $i \neq j$.

Proof. Form a bipartite graph $G$ with vertex set $X \cup \{1, \ldots, m\}$, where an element $x \in X$ is adjacent to $i \in \{1, \ldots, m\}$ if $x \in Y_i$. Condition (*) implies that $G$ satisfies hypothesis (*) of Hall’s marriage theorem, so that there exists a matching $f : \{1, \ldots, m\} \to X$. Now set $y_1 = f(1)$, $y_2 = f(2)$, and so forth.

In the symmetric case $|V_0| = |V_1|$, Question 4 can be regarded as a special case of a more general question.
**Definition 11.** Let $G$ be a graph. A **matching** of $G$ is a set $M$ of edges of $G$, no two of which share a vertex. We say that a matching is **perfect** if every vertex of $G$ belongs to some edge of $M$.

**Question 12.** Given a graph $G$, when does it have a perfect matching?

An answer is provided by the following result:

**Theorem 13** (Tutte). Let $G$ be a finite graph with vertex set $V$. Then $G$ has a perfect matching if and only if the following condition is satisfied, for every subset $S \subseteq V$:

\[(\star) \text{ Let } G' \text{ be the graph obtained from } G \text{ by removing the set } S. \text{ Then the number connected components of } G' \text{ of odd size is } \leq |S|.\]

**Example 14.** When $S = \emptyset$, condition $(\star)$ asserts that $G$ has no components with an odd number of vertices. In particular, this implies that the number of vertices of $G$ is even.

To prove the necessity of condition $(\star)$, let us suppose that $G$ has a perfect matching $M$. Let $S$ be a set of vertices of $G$ and let $G'$ be as in $(\star)$. If $G''$ is a connected component of $G'$ with an odd number of vertices, then $G'$ does not admit a perfect matching. Consequently, there exists at least one edge belonging to $M$ which connects a vertex of $G''$ with one of the vertices of $S$. The vertices of $S$ which arise in this way are all distinct (since two edges of $M$ cannot share a vertex). Consequently, the number of odd components of $G'$ must be $\leq |S|$.

We will prove the sufficiency of condition $(\star)$ in the next lecture.