Let $A$ be a finite partially ordered set. The *incidence matrix* of $A$ is the square matrix $I = [i_{a,b}]_{a,b \in A}$, where

$$i_{a,b} = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise.} \end{cases}$$

In the last lecture, we introduced the M"{o}bius function of $A$. This is a function $\mu : A \times A \rightarrow \mathbb{Z}$ with the following property: the matrix $[\mu(a,b)]_{a,b \in A}$ is an inverse of the incidence matrix $I$.

Our first goal in this lecture is to give a more explicit description of the function $\mu$. First, let us introduce a bit of notation.

**Notation 1.** Let $A$ be a partially ordered set containing elements $a, b \in A$. We let $X_{a,b}$ denote the set of all chains $C \subseteq A$ containing $a$ as a least element and $b$ as a greatest element. In this case, we can write $C = \{a = x_0 < x_1 < \cdots < x_k = b\}$. We will refer to $k$ as the *length* of $C$ and denote it by $l(C)$, so that $l(C) = |C| - 1$.

**Theorem 2.** Let $A$ be a finite partially ordered set. The M"{o}bius function $\mu : A \times A \rightarrow \mathbb{Z}$ is given by the formula

$$\mu(a,b) = \sum_{C \in X_{a,b}} (-1)^{l(C)}.$$

**Proof.** Define $\lambda(a, b) = \sum_{C \in X_{a,b}} (-1)^{l(C)}$. To prove that $\lambda = \mu$, it will suffice to show that the matrix $M = [\lambda(a,b)]_{a,b \in A}$ is an inverse of the incidence matrix $I$. Since $I$ is invertible, it will suffice to show that $MI$ is the identity matrix. Unwinding the definitions, we must show that for $a, c \in A$, the sum

$$\sum_{b \in A} \lambda(b, c) \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise.} \end{cases}$$

is 1 if $a = c$ and zero otherwise. In other words, we wish to show

$$\sum_{b \geq a} \lambda(b, c) = \begin{cases} 1 & \text{if } a = c \\ 0 & \text{if } a \neq c. \end{cases}$$

Invoking the definition of $\lambda_{b,c}$, we can rewrite the right hand side as

$$\sum_{b \geq a} \sum_{C \in X_{b,c}} (-1)^{l(C)}.$$

This can be written as

$$\sum_{C \in Y_{a,c}} (-1)^{l(C)},$$
where \( Y_{a,c} \) denotes the collection of all chains in \( A \) whose largest element is equal to \( c \), and whose smallest element is \( \geq a \).

If \( a = c \), then \( Y_{a,c} \) contains only a single chain \( C = \{c\} \) of length 0, so this sum is equal to 1. Let us therefore assume that \( a \neq c \), and prove that the sum is equal to zero. We divide the set \( Y_{a,c} \) into two parts: let \( Y_+ \subseteq Y_{a,c} \) be the collection of those chains which contain \( a \), and let \( Y_- \) be the collection of those chains which do not. The construction \( C \mapsto C \cup \{a\} \) determines a bijection from \( Y_- \) to \( Y_+ \) (the inverse bijection is given by \( C \mapsto C - \{a\} \)). We can therefore write

\[
\sum_{C \in Y_{a,c}} (-1)^{l(C)} = \sum_{C \in Y_+} (-1)^{l(C)} + \sum_{C \in Y_-} (-1)^{l(C \cup \{a\})}.
\]

On the right hand side, we can cancel the relevant terms pairwise to obtain 0. \( \square \)

**Corollary 3.** Let \( A \) be a finite partially ordered set and \( \mu \) its Möbius function. Then \( \mu \) has the following properties:

1. \( \mu(a, a) = 1 \) for all \( a \in A \).
2. If \( a \not\leq b \), then \( \mu(a, b) = 0 \).

**Proof.** If \( a = b \), then \( X_{a,b} \) consists only of the chain \( C = \{a\} \), so that \( \sum_{C \in X_{a,b}} (-1)^{|l(C)|} = 1 \). This proves (1). To prove (2), note that if \( a \not\leq b \) then \( X_{a,b} \) is empty (there are no chains from \( a \) to \( b \)). \( \square \)

**Corollary 4.** Let \( A \) be a finite partially ordered set and \( \mu \) its Möbius function. Then the definition of \( \mu \) is local. That is, for each \( a, b \in A \), the integer \( \mu(a,b) \) depends only on the partially ordered set \( \{c \in A : a \leq c \leq b\} \).

Theorem 2 can be given a topological interpretation. To every partially ordered set \( A \), one can associate a topological space \( N(A) \), called the *nerve* of \( A \). The space \( N(A) \) is a simplicial complex, whose simplices are given by the chains of \( A \). More precisely, we can construct \( N(A) \) as follows:

- For each \( a \in A \), add a vertex \( v_a \).
- For each \( a < b \) in \( A \), add an edge \( e_{a,b} \) from \( v_a \) to \( v_b \).
- For each \( a < b < c \), add a triangle with vertices \( v_a, v_b, \) and \( v_c \), whose edges are given by \( e_{a,b}, e_{b,c}, \) and \( e_{a,c} \).
- And so forth.

To be still more precise, if we choose an enumeration \( A = \{a_1, \ldots, a_n\} \), then we can define \( N(A) \) to be the subset of \( \mathbb{R}^n \) consisting of those vectors \( (t_1, t_2, \ldots, t_n) \) such that each \( t_i \geq 0 \), \( \sum_{1 \leq i \leq n} t_i = 1 \), and \( \{a_i : t_i \neq 0\} \) is a chain of \( A \).

To any finite simplicial complex \( Y \), one can assign its *Euler characteristic* \( \chi(Y) \). This is simply given by the alternating sum

\[
\sum_{n \geq 0} (-1)^n s_n
\]

where \( s_n \) denotes the number of \( n \)-simplices of \( Y \). In particular, if \( A \) is a partially ordered set, we have

\[
\chi(N(A)) = \sum_{\emptyset \neq C \subseteq A} (-1)^{l(C)},
\]

where the sum is taken over all nonempty chains in \( A \).

Now suppose we are given elements \( a, b \in A \). Assume that \( a < b \) (otherwise, the value of the Möbius function \( \mu(a,b) \) is given by Corollary 3), and set \( A^b_a = \{c \in A : a < c < b\} \). If \( C \) is a chain in \( A \) with
greatest element \( b \) and least element \( a \), then \( C \setminus \{a, b\} \) is a chain in \( A_{\leq a}^b \). Conversely, if \( C \subseteq A_{\leq a}^b \) is a chain, then \( C \cup \{a, b\} \) is a chain in \( A \) with least element \( a \) and greatest element \( b \). Using Theorem 2, we can write

\[
\mu(a, b) = \sum_{C \in X_{a, b}} (-1)^{|C|} = \sum_{D \subseteq A_{\leq a}^b} (-1)^{|D|} (D \cup \{a, b\}) = \sum_{\emptyset \neq D \subseteq A_{\leq a}^b} (-1)^{|D|} = \chi(N(A_{\leq a}^b)) - 1.
\]

Combining this with Corollary 3, we obtain the following:

**Proposition 5.** Let \( A \) be a finite partially ordered set. Then the Möbius function \( \mu : A \times A \to \mathbb{Z} \) is given by

\[
\mu(a, b) = \begin{cases} 
1 & \text{if } a = b \\
\chi(N(A_{\leq a}^b)) - 1 & \text{if } a < b \\
0 & \text{otherwise}.
\end{cases}
\]

**Example 6.** Suppose we elements \( a < b \) of \( A \) which are adjacent, in the sense that there do not exist any elements \( c \) with \( a < c < b \). Then \( A_{\leq a}^b \) is empty, so \( \chi(N(A_{\leq a}^b)) = 0 \), and \( \mu(a, b) = -1 \).

**Example 7.** Let \( A \) be the collection of all subsets of the set \( \{1, 2\} \), ordered by inclusion. Let \( a = \emptyset \) and \( b = \{1, 2\} \) be the least and greatest elements of \( A \), respectively. Then \( A_{\leq a}^b \) is the collection of one-element subsets of \( \{1, 2\} \), which is a two-element antichain. Then the nerve \( N(A_{\leq a}^b) \) consists of two points (with the discrete topology). We get \( \chi(N(A_{\leq a}^b)) = 2 \), so that \( \mu(a, b) = 1 \).

**Example 8.** Let \( A \) be the collection of all subsets of the set \( \{1, 2, 3\} \), ordered by inclusion. Set \( a = \emptyset \) and \( b = \{1, 2, 3\} \). The nerve of \( A_{\leq a}^b \) is a one-dimensional simplicial complex depicted in the diagram

\[
\begin{array}{cccc}
\{1\} & \{1, 2\} & \{1, 3\} \\
| & \uparrow & \uparrow \\
\{2\} & \{3\} \\
\downarrow & \downarrow & \\
\{2, 3\}.
\end{array}
\]

Topologically, this is a circle. It has Euler characteristic 0 (since it has 6 vertices and 6 edges), so we get \( \mu(a, b) = -1 \).

Examples 7 and 8 can be generalized. Let \( A \) be the collection of subsets of the set \( \{1, \ldots, n\} \). If we remove the least and greatest elements \( a, b \in A \), we obtain a new partially ordered set \( A_0 \). One can show that \( N(A_0) \) is a sphere of dimension \( n - 2 \), and therefore has Euler characteristic \( 1 + (-1)^n \). It follows that \( \mu(a, b) = (-1)^n \). However, we would like to prove this more directly, without making a digression into topology.