Recall that species \( S \) is said to be molecular if there is exactly one \( S \)-structure, up to isomorphism. Equivalently, \( S \) is molecular if there exists an integer \( n \) such that \( S[\langle m \rangle] \) is empty for \( m \neq n \), and \( S[\langle n \rangle] \) is acted on transitively by the symmetric group \( \Sigma_n \). Every species \( S \) can be decomposed uniquely as a sum \( \sum \alpha S_\alpha \) of molecular species. Furthermore, there is a one to one correspondence between (isomorphism classes of) molecular species and (isomorphism classes of) pairs \((G, X)\), where \( G \) is a finite group acting faithfully on a finite set \( X \). This correspondence assigns to a pair \((G, X)\) the species \( S(G, X) \), where \( S(G, X)[I] = \text{Bij}(X, I)/G \).

If \( S \) is any species, the cycle index of \( S \) is given by

\[
Z_S(s_1, s_2, \ldots) = \sum_{\rho \in \Sigma_n} Z_{\text{Aut}(I, \eta)}(s_1, \ldots)
\]

where the sum is over all isomorphism classes of \( S \)-structures \((I, \eta)\). If \( G \) is a finite group acting faithfully on a set \( X \) and \( S = S(G, X) \) is the corresponding molecular species, then there is only one isomorphism class of \( S \)-structures, and its automorphism group is given by \( G \). We therefore have

\[
Z_S(s_1, s_2, \ldots) = Z_G(s_1, s_2, \ldots) : \]

in other words, we can regard the cycle index of a group \( G \) as a special case of the cycle index of a species.

**Remark 1.** Let \( G \) be a finite group acting on a set \( X \). The definition of the cycle index \( Z_G(s_1, s_2, \ldots) \) does not require that the action of \( G \) on \( X \) is faithful. However, there is no harm in assuming that. Suppose that we are given an arbitrary action of a finite group \( G \) on a finite set \( X \), given by a map \( \rho : G \rightarrow \text{Perm}(X) \). Let \( N = \ker(\rho) \) be the kernel of \( \rho \); that is, the subgroup of \( G \) consisting of elements which fix every \( x \in X \). Then \( N \) is a normal subgroup of \( G \), and the quotient group \( G/N \) acts on \( X \). Moreover, we have

\[
Z_G(s_1, \ldots) = Z_{G/N}(s_1, \ldots).
\]

Here is a more direct description of the cycle index of a species:

**Proposition 2.** Let \( S \) be a species. Then the cycle index of \( S \) is given by

\[
Z_S(s_1, s_2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} |S[\langle n \rangle]|^{\sigma} s_1^{k_1} s_2^{k_2} \cdots,
\]

where \( k_m \) denotes the number of \( m \)-cycles in the permutation \( \sigma \).

**Proof.** We can rewrite the right hand side as

\[
\sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sum_{\eta \in S[\langle n \rangle]} \begin{cases} s_1^{k_1} s_2^{k_2} \cdots & \text{if } S[\sigma](\eta) = \eta \\ 0 & \text{otherwise.} \end{cases}
\]
Rearranging the order of summation, this is given by
\[
\sum_{n \geq 0} \sum_{\eta \in \mathcal{S}(n)} \frac{1}{n!} \sum_{\sigma \in G} Z_{\sigma},
\]
where \( G = \text{Stab}(\eta) \) denote the stabilizer of the point \( \eta \) and \( Z_{\sigma} \) is the cycle monomial of \( \sigma \) (regarded as a permutation of the set \( \{1, 2, \ldots, n\} \)). We can rewrite this as
\[
\sum_{n \geq 0} \sum_{\eta \in \mathcal{S}(n)} \frac{|G|}{n!} Z_{G}(s_1, \ldots).
\]
Note that the contribution coming from a particular element of \( \mathcal{S}(\langle n \rangle) \) is the same for all other \( \eta' \) belonging to the \( \Sigma_n \) orbit of \( \eta \). The number of elements in this orbit is given by \( \frac{n!}{|G|} \). We may therefore write our sum as
\[
\sum_{n \geq 0} \sum_{\eta \in \mathcal{S}(\langle n \rangle) / \Sigma_n} Z_{G}(s_1, \ldots),
\]
which reproduces the definition of \( Z_S \).

**Example 3.** Let \( S \) be any species, and consider the power series
\[
Z_S(x, 0, 0, \ldots).
\]
Writing
\[
Z_S = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} |S(\langle n \rangle)|^{\sigma} s_1^{k_1} s_2^{k_2} \cdots,
\]
we note that the contribution from any non-identity permutation vanishes. We therefore obtain
\[
Z_S(x, 0, 0, \ldots) = \sum_{n \geq 0} \frac{|S(\langle n \rangle)|}{n!} x^n,
\]
thereby recovering the exponential generating function of \( S \).

**Example 4.** Let \( S \) be any species, and consider the power series \( Z_S(x, x^2, x^3, \ldots) \). This is given by
\[
Z_S(x, x^2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} |S(\langle n \rangle)|^{\sigma} x^n.
\]
Applying Burnside’s formula, we see that this is given by
\[
\sum_{n \geq 0} |S(\langle n \rangle) / \Sigma_n| x^n.
\]
This is the ordinary generating function for the unlabelled enumeration problem of counting \( S \)-structures, up to isomorphism (this formula also follows immediately from Definition ??).

Let’s now compute an example.

**Question 5.** Let \( S \) be the species of sets with no structure: that is, \( S[I] = \{\ast\} \) for every finite set \( I \). What is the cycle index of \( S \)?
According to Proposition 2, the answer is given by

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} s_{k_1}^{s_{k_1}} s_{k_2}^{s_{k_2}} \cdots$$

where $k_i$ denotes the number of $i$-cycles of $\sigma$. We can rewrite this as

$$\sum_{n \geq 0} \frac{1}{n!} \sum_{k_1 + 2k_2 + \cdots = n} C_{k_1} s_{k_1}^{s_{k_1}} s_{k_2}^{s_{k_2}} \cdots,$$

where $C_{\vec{k}}$ denotes the number of permutations having exactly $k_i$ $i$-cycles. Let’s first determine the numbers $C_{\vec{k}}$.

Fix a decomposition $n = k_1 + 2k_2 + 3k_3 + \cdots$. Suppose $\sigma$ is a permutation with $k_1$ 1-cycles, $k_2$ 2-cycles, and so forth. How many possibilities are therefore $\sigma$? First, let’s count the number of ways to partition $\sigma$ into labelled subsets, $k_1$ of which have size 1, $k_2$ of which have size 2, and so forth. This is given by the multinomial coefficient

$$\frac{n!}{(1!)^{k_1} (2!)^{k_2} (3!)^{k_3} \cdots}.$$  

Our counting problem has a slightly different answer. The cycles of $\sigma$ are not labelled, so we must divide by the product $k_1!k_2!\cdots$. Also, a permutation is not determined by the decomposition of $\{1, 2, \ldots, n\}$ into orbits: we must also specify a cyclic permutation of each orbit. Consequently, we should multiply by $(0!)^{k_1} (1!)^{k_2} (2!)^{k_3} \cdots$. We therefore obtain

$$C_{\vec{k}} = \frac{n!}{k_1!k_2!\cdots} \frac{(0!)^{k_1} (1!)^{k_2} (2!)^{k_3} \cdots}{n!} \frac{(1!)^{k_1} (2!)^{k_2} (3!)^{k_3} \cdots}{(k_1!k_2!\cdots)(1^{k_1}2^{k_2}\cdots)}.$$  

Plugging this in, we get

$$Z_S = \sum_{n \geq 0} \frac{1}{n!} \sum_{n = k_1 + 2k_2 + 3k_3 + \cdots} \frac{n!}{(k_1!k_2!\cdots)(1^{k_1}2^{k_2}\cdots)} s_{k_1}^{s_{k_1}} s_{k_2}^{s_{k_2}} \cdots$$

$$= \sum_{k_1, k_2, k_3, \ldots} \prod_{i \geq 1} s_i^{k_i} \frac{1}{k_i!^{k_i}}$$

$$= \prod_{i \geq 1} \sum_{k \geq 0} \frac{1}{k!} \left( \frac{s_i}{i} \right)^k$$

$$= \prod_{i \geq 1} e^{s_i/i}$$

$$= e^{s_1 + s_2/2 + s_3/3 + \cdots}.$$