(1) Let $S$ be an infinite set, and let $\mathbb{R}^S$ denote the product $\prod_{s \in S} \mathbb{R}$, whose elements are functions $f : S \to \mathbb{R}$. Let $X \subseteq \mathbb{R}^S$ be the subset consisting of those functions $f$ such that the set $\{s \in S : f(s) \neq 0\}$ is finite. Show that $X$ is dense in $\mathbb{R}^S$ with respect to the product topology, but not with respect to the box topology.

(2) Let $\{(X_i, d_i)\}_{i \geq 0}$ be a sequence of metric spaces, and suppose that $d_i(x, y) \leq 1$ for all $x, y \in X_i$. Let $X = \prod_{i} X_i$ and define a map $d : X \times X \to \mathbb{R}$ by the formula $d(x, y) = \sum_{i \geq 0} d_i(x_i, y_i)$ (here $x_i$ and $y_i$ denote the images of $x$ and $y$ in $X_i$). Show that $d$ is a metric on $X$, and that the metric topology on $X$ coincides with the product topology (where each $X_i$ is endowed with the metric topology).

(3) A topological space $X$ is said to be metrizable if it coincides with the metric topology for some metric $d : X \times X \to \mathbb{R}$. Show that a countable product of metrizable spaces is metrizable. Show that an uncountable product of metrizable spaces need not be metrizable.

(4) Let $A$ be a linearly ordered set. For every pair of elements $a, b \in A$, we let $(a, b) = \{c \in A : a < c < b\}$, $(-\infty, b) = \{c \in A : c < b\}$, and $(a, \infty) = \{c \in A : a < c\}$. Show that there is a topology on $A$ having as basis the collection of sets of the form $A$, $(a, b)$, $(-\infty, b)$, and $(a, \infty)$, for $a, b \in A$. This topology is called the order topology on $A$.

(5) Let $A$ be any linearly ordered set, and regard $A$ as a topological space with respect to the order topology of problem (4). Show that $A$ is Hausdorff. Show that if $A$ is well-ordered and has a largest element, then $A$ is compact.