Math 131: Problem Set 1

(1) Let \((X, d)\) and \((X', d')\) be metric spaces. We say that a function \(f : X \rightarrow X'\) is Lipschitz if there exists a constant \(c > 0\) such that \(d'(f(x), f(y)) \leq cd(x, y)\). Show that any Lipschitz function between metric spaces is continuous. Give an example of a continuous function which is not Lipschitz.

(2) We define three notions of distance on \(\mathbb{R}^2\), using the formulas
\[
d((x, y), (x', y')) = \sqrt{(x-x')^2 + (y-y')^2}
\]
\[
d'((x, y), (x', y')) = |x-x'| + |y-y'|
\]
\[
d' '((x, y), (x', y')) = \sup\{|x-x'|, |y-y'|\}.
\]
Show that each of these distance functions endows \(\mathbb{R}^2\) with the structure of a metric space, and that all three metrics define the same topology on \(\mathbb{R}^2\).

(3) Regard \(\mathbb{R}\) as a metric space with respect to the usual notion of distance, given by \(d(x, x') = |x - x'|\). Show that every isometry \(f : \mathbb{R} \rightarrow \mathbb{R}\) has the form \(f(x) = \pm x + \lambda\), for some real number \(\lambda\).

(4) Let \(X\) be a metric space. For every point \(x \in X\) and every real number \(\epsilon > 0\), we let \(B_{< \epsilon}(x) = \{y \in X : d(x, y) < \epsilon\}\) and \(B_{\leq \epsilon}(x) = \{y \in X : d(x, y) \leq \epsilon\}\). Show that \(B_{< \epsilon}(x)\) is an open subset of \(X\) and that \(B_{\leq \epsilon}(x)\) is a closed subset of \(X\). We refer to \(B_{< \epsilon}(x)\) as the open ball around \(x\) of radius \(\epsilon\), and \(B_{\leq \epsilon}(x)\) as the closed ball around \(x\) of radius \(\epsilon\). Give an example to show that \(B_{\leq \epsilon}(x)\) need not be the closure of \(B_{< \epsilon}(x)\).

(5) Give an example to show that a continuous bijection \(f : X \rightarrow X'\) between metric spaces need not be a homeomorphism. In other words, show that the inverse function \(f^{-1}\) need not be continuous.