(1) Let $X$ be a completely regular topological space. Let $A$ and $B$ be closed subsets of $X$ with $A \cap B = \emptyset$, and suppose that $A$ is compact. Show that there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(a) = 0$ for $a \in A$ and $f(b) = 1$ for $b \in B$.

Since $X$ is completely regular, we can choose for each $a \in A$ a continuous function $f_a : X \rightarrow [0, 1]$ such that $f_a(a) = 0$ and $f_a(b) = 1$ for $b \in B$. Let $U_a = f_a^{-1}[0, \frac{1}{2})$. Since $f_a$ is continuous, $U_a$ is an open subset of $X$. This open subset contains $a$, so that $A \subseteq \bigcup_{a \in A} U_a$. Since $A$ is assumed to be compact, there exists a finite subset $\{a_1, \ldots, a_n\} \subseteq A$ such that $A \subseteq U_{a_1} \cup \cdots \cup U_{a_n}$. We define a function $f' : X \rightarrow [0, 1]$ by the formula $f'(x) = \min\{f_{a_i}(x)\}_{1 \leq i \leq n}$, and $f : X \rightarrow [0, 1]$ by the formula $f(x) = \max\{0, 2f'(x) - 1\}$. If $b \in B$, then each $f_{a_i}(b)$ is equal to 1, so that $f(b) = f'(b) = 1$. If $a \in A$, then $a \in U_{a_i}$ for some index $i$, so that $f'(a) \leq f_{a_i}(a) < \frac{1}{2}$ and therefore $f(a) = 0$.

(2) Let $X$ be a normal topological space and let $Y \subseteq X$ be a closed subspace. Show that every continuous map $f_0 : Y \rightarrow \mathbb{R}$ can be extended to a continuous map $f : X \rightarrow \mathbb{R}$.

Choose a homeomorphism $h : \mathbb{R} \rightarrow (-1, 1)$, and regard $h \circ f_0$ as a continuous map from $Y$ into $[-1, 1]$. We proved in class that $h \circ f_0$ can be extended to a map $g : X \rightarrow [-1, 1]$. Then $g^{-1}\{-1, 1\}$ is a closed subset $A \subseteq X$ which is disjoint from $Y$. By the Urysohn Lemma, we can choose a continuous function $g' : X \rightarrow [0, 1]$ such that $g(y) = 1$ for $y \in Y$ and $f(a) = 0$ for $a \in A$. Let $g'' : X \rightarrow [-1, 1]$ be defined by the formula $g''(x) = g(x)g'(x)$. Then $g''(y) = g(y) = (h \circ f_0)(y)$ for $y \in Y$. Moreover, $g''(x) \in (-1, 1)$ for all $x \in X$: if $x \in A$, then $g''(x) = 0$, and if $x \notin A$ then $|g''(x)| = |g(x)|g'(x) < g'(x) \leq 1$. Set $f = h^{-1} \circ g''$. Then $f$ is a continuous extension of $f_0$.

Let $X$ be a Hausdorff topological space. We let $K(X)$ denote the collection of all compact subsets of $X$. We regard $K(X)$ as a topological space, taking as a subbasis all sets of the form

$$\{K \in K(X) : K \subseteq U\} \quad \{K \in K(X) : K \cap U \neq \emptyset\}$$

where $U$ is an open subset of $X$.

(3) Prove that that $K(X)$ is a Hausdorff space and that $K(X)$ is compact if and only if $X$ is compact.
We first show that $K(X)$ is Hausdorff. Let $K, K' \in K(X)$ be distinct compact sets. Without loss of generality there exists a point $x \in K$ such that $x \notin K'$. Choose disjoint open sets $U, V \subseteq X$ such that $x \in U$, $K' \subseteq V$. Then

$$\{Y \in K(X) : Y \cap U \neq \emptyset\} \cup \{Y \in K(X) : Y \subseteq V\}$$

are disjoint open subsets of $K(X)$ containing $K$ and $K'$, respectively.

Now suppose that $K(X)$ is compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of $X$; we will show that it admits a finite subcover. For every finite subset $A_0 \subseteq A$, let $U_{A_0} = \bigcup_{\alpha \in A_0} U_\alpha$, and let $V_{A_0} = \{K \in K(X) : K \subseteq U_{A_0}\}$. If $K \in K(X)$, then $\{U_\alpha \cap K\}$ is an open covering of $K$; the compactness of $K$ then guarantees the existence of a finite subcover which shows that $K \subseteq U_{A_0}$ for some finite subset $A_0 \subseteq A$. It follows that $K(X) = \bigcup_{A_0} V_{A_0}$. Since $K(X)$ is compact, there exists a finite subcover: that is, there is a finite list of finite subsets $A_1, A_2, A_3, \ldots, A_n \subseteq A$ such that $K(X) = V_{A_1} \cup \cdots \cup V_{A_n}$. Let $A' = A_1 \cup \cdots \cup A_n$; we will show that $X = \bigcup_{\alpha \in A'} U_\alpha$. To see this, we note that for each $x \in X$ the subset $\{x\} \subseteq X$ is compact, so that $\{x\} \in V_{A_i}$ for some $i$ and therefore $x \in U_\alpha$ for some $\alpha \in A_i$.

Finally, suppose that $X$ is compact. We will show that $K(X)$ is compact. It suffices to show that every open covering of $K(X)$ by subbasic open sets has a finite subcovering. To this end, suppose that we are given collections of open sets $\{U_\alpha\}_{\alpha \in A}, \{V_\beta\}_{\beta \in B}$ in $X$ such that

$$K(X) = \left( \bigcup_{\alpha \in A} \{K \in K(X) : K \subseteq U_\alpha\} \right) \cup \left( \bigcup_{\beta \in B} \{K \in K(X) : K \cap V_\beta \neq \emptyset\} \right).$$

Let $V = \bigcup_{\beta} V_\beta$ and let $K = X - V$. Then $K$ is closed in $X$ and therefore compact. Since $K \notin \bigcup_{\beta \in B} \{K \in K(X) : K \cap V_\beta \neq \emptyset\}$, we must have $K \subseteq U_\alpha$ for some $\alpha \in A$. Let $K' = X - U_\alpha$. Then $K' \subseteq V$. Since $K'$ is a closed subset of $X$, it is compact. It follows that there exists a finite subset $B_0 \subseteq B$ such that $K' \subseteq \bigcup_{\beta \in B_0} V_\beta$. Then every subset of $X$ is either contained in $U_\alpha$ or intersects $V_\beta$ for some $\beta \in B_0$, so that

$$K(X) = \{K \in K(X) : K \subseteq U_\alpha\} \cup \left( \bigcup_{\beta \in B_0} \{K \in K(X) : K \cap V_\beta \neq \emptyset\} \right)$$

(4) Let $X$ be a Hausdorff space. Show that the map $f : K(X) \times K(X) \to K(X)$ given by $(K, K') \mapsto K \cup K'$ is continuous. Give an example to show that the map $g : K(X) \times K(X) \to K(X)$ given by $(K, K') \mapsto K \cap K'$ need not be continuous.

We first show that $f$ is continuous. It will suffice to show that for every subbasic open set $W \subseteq K(X)$, the inverse image $f^{-1}(W)$ is an open subset of $K(X) \times K(X)$. If $W = \{K \in K(X) : K \subseteq U\}$, then $f^{-1}(W) = W \times W$. If $W = \{K \in K(X) : K \cap V \neq \emptyset\}$, then $f^{-1}(W) = (K(X) \times W) \cup (W \times K(X))$.

We now show that $g$ is not continuous in the case $X = \mathbb{R}$. Consider the sequence of sets $\{\frac{1}{n}\}$. We claim that this sequence converges to $\{0\}$ in $K(\mathbb{R})$. 
Let $W$ be an open subset of $K(\mathbb{R})$ containing $\{0\}$; we claim that $W$ contains $\{\frac{1}{n}\}$ for all but finitely many $n$. Without loss of generality, we may assume that $W$ is a finite intersection of subbasic open sets, and is therefore of the form

$$\{K \in K(X) : (K \subseteq U) \land (K \cap V_1 \neq \emptyset) \land \cdots \land (K \cap V_n \neq \emptyset)\}$$

for some open sets $U, V_1, \ldots, V_n \subseteq X$. Since $\{0\} \in W$, we have $0 \in U \cap V_1 \cap \cdots \cap V_n$. Since the intersection is open, it must contain the interval $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. It follows that $\frac{1}{n} \in U \cap V_1 \cap \cdots \cap V_n$ and therefore $\{\frac{1}{n}\} \in W$ whenever $\frac{1}{n} < \epsilon$.

The sequence of points $\{(\{\frac{1}{n}\}, \{0\})\}_{n>0}$ converges to $\{(0), 0\}$ in $K(X) \times K(X)$ if $g$ were continuous, it would follow that the sequence $\{g(\{\frac{1}{n}\}, \{0\})\}_{n>0}$ converges to $g(\{0\}, \{0\}) = \{0\} \in K(X)$. But $\{\frac{1}{n}\} \cap \{0\} = \emptyset$, so the sequence $\{g(\{\frac{1}{n}\}, \{0\})\}_{n>0}$ converges to $\emptyset$. Since $\emptyset \neq \{0\}$, we obtain a contradiction (note that $K(X)$ is Hausdorff, so convergent sequences in $K(X)$ can have at most one limit).

(5) Let $Z$ be the set of integers, endowed with the discrete topology. Let $Z^\vee$ denote the Stone-Cech compactification of $Z$. Prove that $Z^\vee$ is uncountable.

Let $S$ be the set of rational numbers in the interval $[0, 1]$. Since $S$ is countable, we can choose a surjective map $f : Z \to S$. We can regard $f$ as a continuous map from $Z$ to $[0, 1]$. Since $[0, 1]$ is a compact Hausdorff space, $f$ extends to a continuous map $f^\vee : Z^\vee \to [0, 1]$. Then $f^\vee(Z^\vee)$ is continuous image of a compact set, and therefore a compact subset $K \subseteq [0, 1]$. Then $K$ is a closed subset of $[0, 1]$ containing $f(Z) = S$. Since $S$ is dense, we have $K = [0, 1]$. Since $[0, 1]$ is uncountable, the existence of a surjection $f^\vee : Z^\vee \to [0, 1]$ guarantees that $Z^\vee$ is uncountable.

(6) Let $X$ be a compact Hausdorff space, and let $x, y \in X$. Assume that for any subset $U \subseteq X$ which is both closed and open, either $x, y \in U$ or $x, y \notin U$. Show that there exists a connected subset of $X$ which contains both $x$ and $y$. (For some hints, see problem 4 in §5-37 of the textbook).

Let $\{U_\alpha\}_{\alpha \in A}$ be the collection of all subsets $U$ of $X$ which are both closed and open such that $x \in U$. Let $S = \bigcap_\alpha U_\alpha$. Then $x \in S$ by construction, and our hypothesis guarantees also that $y \in S$. We will show that $S$ is connected. Suppose otherwise. Then we can write $S$ as a disjoint union of nonempty closed subsets $S', S'' \subseteq S$. The sets $S'$ and $S''$ are closed in $X$ and therefore compact. It follows that we can find disjoint open sets $V, W \subseteq X$ such that $S' \subseteq V$ and $S'' \subseteq W$. Then

$$X = V \cup W \cup (\bigcup_{\alpha \in A_0} (X - U_\alpha))$$

Since $X$ is compact, there exists a finite subset $A_0 \subseteq A$ such that $X = V \cup W \cup (\bigcup_{\alpha \in A_0} (X - U_\alpha))$. Let $U = \bigcap_{\alpha \in A_0} U_\alpha$. Then $U$ is a closed and open subset of $X$ containing $x$ and $y$, and $U \subseteq V \cup W$. It follows that $U$ is a disjoint union of
open sets \((U \cap V)\) and \((U \cap W)\), so that the sets \(U \cap V\) and \(U \cap W\) are also both closed and open. Without loss of generality we may assume that \(x \in U \cap V\). Then \(U \cap V = U_\alpha\) for some \(\alpha \in A\), so that \(S \subseteq U \cap V \subseteq V\) and therefore \(S' = S \cap V = S\). Then \(S'' = S - S'\) is empty, contrary to our assumption.