Math 114, Problem Set 7 (due Monday, November 4)

November 1, 2013

(1) Let $V_0, V_1, V_2, V_3, \ldots$ be real vector spaces with norms $\| \cdot \|_n : V_n \to \mathbb{R}_{\geq 0}$.

Given an element $\vec{v} = (v_n)_{n \geq 0} \in \prod_{n \geq 0} V_n$, let

$$\| \vec{v} \| = \sum_{n \geq 0} \| v_n \|_n \in \mathbb{R}_{\geq 0} \cup \{ \infty \}.$$  

Let $V \subseteq \prod_{n \geq 0} V_n$ be the subset consisting of those elements $\vec{v}$ such that $\| \vec{v} \| < \infty$. Show that $V$ is a real vector space and that $\vec{v} \mapsto \| \vec{v} \|$ is a norm on $V$. If each $V_n$ is a Banach space, show that $V$ is a Banach space. We will refer to $V$ as the $\ell^1$-sum of the Banach spaces $\{V_n\}_{n \geq 0}$.

(2) Suppose we are given a sequence $E_0, E_1, E_2, \ldots \subseteq \mathbb{R}^m$ of pairwise disjoint measurable subsets of $\mathbb{R}^m$. Let $E = \bigcup E_n$. Show that $L^1(E)$ is isomorphic to the $\ell^1$-sum of the Banach spaces $L^1(E_n)$.

(3) Let $E \subseteq \mathbb{R}^n$ be a measurable set, let $p$ and $q$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and suppose that $f \in L^p(E)$, $g \in L^q(E)$ are functions satisfying

$$\int_E fg = \| f \|_{L^p} \| g \|_{L^q}.$$  

Prove that either $f = 0$, or there exists a nonnegative real number $\lambda$ such that $|g| = \lambda |f|^{p/q}$ almost everywhere.

(4) Let $p, q, r > 1$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $E \subseteq \mathbb{R}^n$ be measurable, and let $f \in L^p(E)$ and $g \in L^q(E)$. Show that the product function $fg$ belongs to $L^r(E)$, and that

$$\| fg \|_{L^r} \leq \| f \|_{L^p} \| g \|_{L^q}.$$  
