PROBLEM SET VIII: PROBLEMS III, IV

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Problem 1. Let \( f : V \to W \) be a linear map between normed vector spaces. Show that if \( V \) is finite-dimensional, then \( f \) is continuous.

Proof. Let \( \{ e_1, \ldots, e_n \} \) be a basis for \( V \). Then
\[
\| f(v) \|_W = \| f(v_1 e_1 + \cdots + v_n e_n) \|_W = \| v_1 f(e_1) + \cdots + v_n f(e_n) \|_W
\]
\[
\leq |v_1| \| f(e_1) \|_W + \cdots + |v_n| \| f(e_n) \|_W \leq \sum_{k=1}^n |v_k| \left( \max_{1 \leq k \leq n} \| f(e_k) \|_W \right).
\]
Define the 1-norm, \( \| \cdot \|_1 : V \to \mathbb{R} \), by
\[
\| v \|_1 = \sum_{k=1}^n |v_k|, \quad \text{with} \quad v = \sum_{k=1}^n v_k e_k.
\]
It suffices to show that \( \| v \|_1 \leq \alpha \| v \|_V \) for arbitrary constant \( \alpha \) and \( V \) a finite dimensional vector space. We have the following
\[
\| v \|_V = \left\| \sum_{k=1}^n v_k e_k \right\|_V \leq \sum_{k=1}^n |v_k| \| e_k \|_V \leq \left( \max_{1 \leq k \leq n} \| e_k \|_V \right) \| v \|_1.
\]
This implies \( \| v \|_V \leq \beta \| v \|_1 \) for some constant \( \beta \). Thus, our norm on \( V \), \( \| \cdot \|_V : V \to \mathbb{R} \), is continuous with respect to the topology induced by the 1-norm. Taking \( \| v - w \|_1 \leq \varepsilon \), we see that
\[
\| v \|_V - \| w \|_V \leq \| v - w \|_V \leq M \varepsilon,
\]
by the Triangle Inequality. Consider the compact set \( \Omega = \{ v \in V : \| v \|_1 = 1 \} \). By the compactness of our set and the continuity of the norm on \( V \), \( \| \cdot \|_V \) achieves a minimum on \( \Omega \). Denoting this minimum by \( \xi \), we have \( 0 < \xi \leq \| v \|_V \), for any \( v \in V \) where \( \| v \|_1 = 1 \). Thus, \( \xi \| v \|_1 \leq \| v \|_V \). Selecting our constant \( \alpha \) appropriately yields our desired result. \( \square \)

Problem 2. Let \( P \) be a partially ordered set and suppose that every linearly ordered subset of \( P \) has an upper bound. Prove that \( P \) has a maximal element by completing the argument outlined in class. Assume (for a contradiction) that \( P \) has no maximal element.

(a) Show that for each linearly ordered subset \( Q \subseteq P \), there exists an element \( \lambda (Q) \in P \) such that \( q < \lambda (Q) \) for each \( q \in Q \).

We will say that a subset \( Q \subseteq P \) is a good chain if \( Q \) is well-ordered and each element \( x \in Q \) satisfies the formula \( x = \lambda \{ q \in Q : q < x \} \).

(b) Show that there is no largest good chain in \( P \).

(c) Show that if \( Q \) and \( Q' \) are good chains, then exactly one of the following conditions holds: (i) \( Q = Q' \); (ii) There exists an element \( q_0 \in Q \) such

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that $Q' = \{ q \in Q : q < q_0 \}$; (iii) There exists an element $q'_0 \in Q'$ such that $Q = \{ q' \in Q' : q' < q'_0 \}$.

(d) Show that if $\{ Q_\alpha \}$ is a collection of good chains, then the union $\bigcup Q_\alpha$ is also a good chain.

(e) Find a contradiction between (b) and (d).

Proof. (a) Let $x \in Q$ be an upper bound for $Q$. Since $x$ is not maximal, there exists some $\hat{x} \in P$ such that $\hat{x} > x$ (by the Axiom of Choice). Set $\lambda (Q) = \hat{x}$, so that for any $y \in Q$, we have $y \leq x < \lambda (Q)$.

(b) Let us assume that $Q$ is the largest good chain in $P$. Let $Q^+ = Q \cup \{ \lambda (Q) \}$. For each $x \in Q$, we have that $x = \lambda (\{ q \in Q^+ : q < x \})$, since $\lambda (Q) > x$ for all $x \in Q$. Definitionally, $\lambda (Q) = \lambda (\{ q \in Q^+ : q < x \})$, so $Q^+$ is a good chain.

(c) If $Q = Q^'$ we are done. Thus, let us suppose that $Q, Q'$ are good chains and $Q \neq Q'$. We show that either $Q \subset Q'$ or $Q' \subset Q$. Let $\Lambda$ be the union of all subsets of $P$ such that $P \subseteq Q$ and $P \subseteq Q'$. Then $\Lambda$ is the largest such set, which is also well-ordered by the well-ordering of $Q$ and $Q'$. Suppose that $\Lambda \neq Q$ and $\Lambda \neq Q'$. Then we may select $q \in Q$ and $q' \in Q'$ such that $q$ is the minimal element of $Q$ and $q \notin S$. Define $q'$ analogously. Thus, $\Lambda \subseteq \{ x \in Q : x < q \}$ and $\Lambda \subseteq \{ x \in Q' : x < q' \}$. Then let $x_m$ be the smallest element of $Q$ such that $x_m < q$ and $x_m \notin \Lambda$. Then we have

$$\{ x \in Q : x < x_m \} \subseteq \Lambda \cup \{ x \in Q' : x < q' \} \subseteq Q.$$

Thus, $x_m = \lambda (\{ x \in Q : x < x_m \}) \subseteq Q'$ so $x_m \in Q \cap Q'$. However, $x_m$ could have been appended to $\Lambda$, thereby contradicting the maximality of $\Lambda$. Thus

$$\Lambda = \{ x \in Q : x < q \} = \{ x \in Q' : x < q' \} \Rightarrow \lambda (\Lambda) = q = q'.$$

However, we could have appended $q$ to $\Lambda$, again contradicting the maximality of $\Lambda$. Thus, either $Q \subset Q'$ or $Q' \subset Q$.

WLOG suppose that $Q \subset Q'$, and let $q' \in Q'$ be the smallest element of $Q'$ such that $q' \notin Q$. We have $\{ x \in Q' : x < q' \} \subseteq Q$. Suppose that $Q \neq \{ x \in Q' : x < q' \}$, and let $x_m$ be the smallest element in $Q$ such that $x_m > q'$. However, by definition and total ordering, we have

$$x_m = \lambda (\{ x \in Q : x < x_m \}) = \lambda (\{ x \in Q : x < q' \}) = q'.$$

Contradiction. Thus, $Q = \{ x \in Q' : x < q' \}$. The second case follows analogously if $Q' \subset Q$.

(d) Let $\{ Q_\alpha \}$ be a collection of good chains. From (c), we know that for any two elements $Q_\theta, Q_\omega$ of $\{ Q_\alpha \}$, we have $Q_\theta \subset Q_\omega, Q_\omega \subset Q_\theta$, or $Q_\theta = Q_\omega$. Thus, the union of any number of good chains will be equal to a good chain, and $\bigcup Q_\alpha$ is a good chain.

(e) The above implies that $\bigcup Q_\alpha$ is the largest good chain. This contradicts (b), so we conclude that $P$ has a maximal element. \qed