PROBLEM SET VI SOLUTIONS (3,4)

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Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f (x, y) = \begin{cases} 1 & \text{if } (\exists n \in \mathbb{Z}_{\geq 0}) [n \leq x, y < n + 1] \\ -1 & \text{if } (\exists n \in \mathbb{Z}_{\geq 0}) [n \leq x < n + 1 \leq y < n + 2] \\ 0 & \text{otherwise.} \end{cases}$$

For each $x \in \mathbb{R}$, let $f_x$ denote the function given by $f_x (y) = f (x, y)$. For each $y \in \mathbb{R}$, let $f_y$ denote the function given by $f_y (x) = f (x, y)$. Show that the functions $x \mapsto \int_{\mathbb{R}} f_x$, $y \mapsto \int_{\mathbb{R}} f_y$ are integrable, and compute their integrals (in other words, compute the double integrals $\int (\int f (x, y) \, dx) \, dy$ and $\int (\int f (x, y) \, dy) \, dx$). Why does the result not contradict Fubini’s Theorem?

Proof. We may write

$$f_x (y) = \begin{cases} 1, & x \geq 0 \text{ and } |x| \leq |x| + 1, \\ -1, & x \geq 0 \text{ and } |x| + 1 \leq |x| + 2, \\ 0, & \text{else,} \end{cases}$$

and

$$f_y (x) = \begin{cases} 1, & y \geq 0 \text{ and } |y| \leq |y| + 1, \\ -1, & y \geq 1 \text{ and } |y| - 1 \leq |y|, \\ 0, & \text{else.} \end{cases}$$

From this it is clear that

$$\int_{\mathbb{R}} f_x = 0, \quad \int_{\mathbb{R}} f_y = \begin{cases} 1, & y \in [0, 1), \\ 0, & \text{else.} \end{cases}$$

Thus, we have that

$$\int \left( \int f (x, y) \, dx \right) \, dy = 0, \quad \int \left( \int f (x, y) \, dy \right) \, dx = 1.$$

This does not contradict Fubini’s Theorem because $f (x, y)$ is not an integrable function. Explicitly, observe that $|f| \geq \chi_{\mathcal{S}}$ where $\mathcal{S} = \{(x, y) : n \leq x, y < n + 1, n \in \mathbb{Z}_{\geq 0}\}$, and $\mu (\mathcal{S}) = \infty$. The measure of $\mathcal{S}$ is infinite because

$$\mathcal{S} = \bigcup_{n=0}^{\infty} \{(x, y) : n \leq x, y < n + 1\}.$$
and each \( \{(x, y) : n \leq x, y < n + 1\} \) has measure one. Therefore,
\[
\int_{\mathbb{R}^2} |f| \geq \int_{\mathbb{R}^2} \chi_{\mathcal{F}} = \mu(\mathcal{F}) = \infty.
\]
\[\square\]

**Problem 2.** Let \( \phi : \mathbb{R} \to \mathbb{R} \) be the continuous function satisfying the inequality
\[
\phi \left( \frac{x + y}{2} \right) \leq \frac{\phi(x) + \phi(y)}{2}
\]
for all \( x, y \in \mathbb{R} \). Show that \( \phi \) is convex; that is, for each real number \( \lambda \in [0, 1] \), we have
\[
\phi (\lambda x + (1 - \lambda) y) \leq \lambda \phi (x) + (1 - \lambda) \phi (y)
\]
for all \( x, y \in \mathbb{R} \).

**Proof.** We will demonstrate the convexity condition for every dyadic \( \lambda = \alpha/2^i \in [0, 1] \) by induction on the power of two in the denominator. We have the base case by assumption. Now suppose that convexity holds for dyadic \( \lambda \) with the exponent in the denominator \( i \) or less. Then, taking \( \alpha = 2^i + 1 \) to be odd, we have
\[
\phi \left( \left( \frac{\alpha}{2^i} \right) x + \left( 1 - \frac{\alpha}{2^i} \right) y \right)
\]
\[
\leq \frac{1}{2} \left( \phi \left( \left( \frac{\beta}{2^i} \right) x + \left( 1 - \frac{\beta}{2^i} \right) y \right) + \phi \left( \left( \frac{\beta + 1}{2^i} \right) x + \left( 1 - \frac{\beta + 1}{2^i} \right) y \right) \right)
\]
\[
\leq \frac{1}{2} \left( \left( \frac{\beta}{2^i} + \frac{\beta + 1}{2^i} \right) \phi(x) + \left( 2 - \frac{\beta + 1}{2^i} \right) \phi(y) \right)
\]
\[
= \left( \frac{\alpha}{2^i} + \frac{\beta + 1}{2^i} \right) \phi(x) + \left( 1 - \frac{\alpha}{2^i} \right) \phi(y).
\]
More generally, suppose that \( \lambda \in [0, 1] \), and select a convergent sequence \( \{\alpha_i/2^i\} \to \lambda \). Fix \( x, y \in \mathbb{R} \), and let \( \epsilon > 0 \). Then, let
\[
f(z) = \phi \left( zx + (1 - z) y \right).
\]
By continuity and convergence, there exists some \( n > 0 \) such that
\[
\left| \frac{\alpha_n}{2^n} - \lambda \right| < \frac{\epsilon}{2 \max |\phi(x)|, |\phi(y)|}
\]
and
\[
f \left( \frac{\alpha_n}{2^n} \right) - f(\lambda) \leq \frac{\epsilon}{2}.
\]
Thus,
\[
f(\lambda) \leq f \left( \frac{\alpha_n}{2^n} \right) + \frac{\epsilon}{2} \leq \frac{\alpha_n}{2^n} \phi(x) + \left( 1 - \frac{\alpha_n}{2^n} \right) \phi(y) + \frac{\epsilon}{2} \leq \lambda \phi(x) + (1 - \lambda) \phi(y) + \epsilon.
\]
Since our \( \epsilon \) was arbitrarily chosen, we are done. \[\square\]