PROBLEM SET X: PROBLEMS (3, 4)

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Problem 1. Let \( V \) be a Banach space. Show that the dimension of \( V \) is either finite or uncountable (that is, \( V \) does not have a countably infinite basis).

Proof. Let \( V \) be a Banach space, and suppose that it has a countably infinite basis \( \{v_i\}_{i>0} \). Next, consider the subsets \( V_n = V(v_1, \ldots, v_n) \), which are finite dimensional subspaces of \( V \). It is clear that each subspace is closed in \( V \).

Now we show that \( V_n \) is nowhere dense in \( V \). Consider any \( \alpha \in V_n \) and \( \epsilon > 0 \). Then the open ball \( B_\epsilon(\alpha) \) contains the element

\[
\left( \alpha + \frac{\epsilon}{2 \|v_{n+1}\|} v_{n+1} \right),
\]

which is not an element of \( V_n \). Hence, each \( V_n = \overline{V_n} \) is nowhere dense in \( V \).

Since each \( V_n \) is closed and nowhere dense in \( V \), the complements, \( (V_n)^c \), are open and dense in \( V \). Applying the Baire Category Theorem, the set \( \bigcap_{n>0} (V_n)^c \) is dense in \( V \), i.e.,

\[
\left( \bigcap_{n>0} (V_n)^c \right)^c = \bigcup_{n>0} V_n = V
\]
is nowhere dense in \( V \). Thus we have a contradiction, and we are done. \( \square \)

Problem 2. Let \( E \subseteq \mathbb{R}^n \) be a measurable set with \( 0 < \mu(E) < \infty \). Let us regard \( L^1(E) \) as a metric space, and \( L^2(E) \) as a subset of \( L^1(E) \). Show that \( L^2(E) \) is meagre (that is, it is a countable union of nowhere dense subsets of \( L^1(E) \)).

Proof. Consider the sets

\[
\mathcal{S}_n = \left\{ f \in L^1(E) : \int_E |f|^2 > n \right\}
\]

and let \( (\mathcal{S}_n)^c \) be its complement. Then we may write

\[
L^2(E) = \bigcup_n (\mathcal{S}_n)^c,
\]
as every element of \( L^2(E) \) must have square integral bounded by \( n \). It suffices to prove that \( \mathcal{S}_n \) is dense and open for every \( n \), as this implies that each \( (\mathcal{S}_n)^c \) is nowhere dense.

Let \( h \in L^1(E)/\mathcal{S}_n \) such that

\[
\int_E |h|^2 \leq n.
\]

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We wish to show that $h$ is in the closure of $S_n$ with respect to the $L^1$-norm. That is, there is a sequence $\{f_k\}$ in $S_n$ such that

$$\lim_{k \to \infty} \|f_k - h\|_{L^1} = 0.$$ 

Since we have $\mu(E) > 0$, $E$ has a sequence of subsets $\{E_k\}$ such that $\mu(E_k) = \mu(E) / (k^{3/2})$. Define

$$f_k = \begin{cases} 
  h + \sqrt{n} \cdot \sqrt{k} & \text{for } x \in E_k, \\
  h & \text{for } x \in E - E_k.
\end{cases}$$

Then we have $f_k \in L^1(E)$ since

$$\int_E |f_k| = \int_E |h| + \sqrt{n} \cdot \left( \mu(E) / \sqrt{k} \right).$$

Since $k^{-3/2} \to 0$, we have $f_k \to h$ with respect to the $L^1$-norm. However,

$$\int_E |f_k|^2 = \int_E |h|^2 + \left( \mu(E) \sqrt{n} / \sqrt{k} \right) \int_E |h| + n \left( \mu(E) \sqrt{k} \right) > n.$$ 

Thus, $f_k \in S_n$. Hence, $S_n$ dense in $L^1(E)$.

Now we show the openness of $S_n$ to complete the proof. Consider $f$ bounded. For any $f \in S_n$, write

$$f_\xi = \begin{cases} 
  f & \text{for } |f| < \xi, \\
  0 & \text{else}.
\end{cases}$$

Then there exists some $\xi$ such that

$$\int_E |f_\xi|^2 > n \implies \int_E |f|^2 > n.$$ 

Consider the support of $f_\xi$. For all $h \in L^1(E)$ such that $\|h|_{supp(f_\xi)} - h\|_{L^1} < \epsilon$, we will have $h|_{supp(f_\xi)} \in S_n$. This allows us to demonstrate

$$\|h - f\|_{L^1} < \epsilon \implies \|h|_{supp(f_\xi)} - h\|_{L^1} < \epsilon \implies h|_{supp(f_\xi)} \in S_n \implies h \in S_n.$$ 

Thus, we replace $f$ by $f_\xi$ and $E$ by $supp(f_\xi)$. That is, we take $f$ bounded by $\xi$.

Finally, write $h$ such that $\|h - f\| < \epsilon$, hence

$$\int_E |h|^2 \geq \int_E |f|^2 - 2 \int_E |f (h - f)| + \int_E |h - f|^2 \geq \int_E |f|^2 - 2 \int_E |f (h - f)| > n - 2 \xi \epsilon.$$ 

Thus, we have $S_n$ is dense and open for every $n$, which implies that

$$L^2(E) = \bigcup_n (S_n)^c$$

is nowhere dense, i.e., $L^2(E)$ is meagre. 

$\square$