Problem Three.

There were lots of interesting solutions to this problem! I think the slickest is this. Decompose $f$ into $f(x) = f_+(x) - f_-(x)$ in the usual way: $f_+(x) := f|_{E_+}$, where $E_+ = \{ x \in \mathbb{R}^n : f(x) > 0 \} \subset \mathbb{R}^n$ and $E_- = \{ x \in \mathbb{R}^n : f(x) \leq 0 \} \subset \mathbb{R}^n$. By the measurability of $f$, both $E_+$ and $E_-$ are measurable.

Let’s focus our attention on $f_+ : E_+ \rightarrow \mathbb{R}$. By the result of the previous pset, we know that if $\int_{E_+} f_+ = 0$, then $f_+ = 0$ a.e. We can then do the same thing for $E_-$. So it will suffice to show that $\int_{E_+} f_+ = 0$.

Measurable sets can be decomposed into the union of an $F_\sigma$ set and a set of measure zero. So we can write $E_+ = F \cup S$, where $F$ is $F_\sigma$ and $m(S) = 0$, and

$$\int_{E_+} f_+ = \int_{E_+} f = \int_{F} f + \int_{S} f = \int_{F} f.$$

But $F$ is obtained as the countable union of closed sets. Closed sets are complements of open sets. Now open sets in $\mathbb{R}^n$ can be expressed as the countable union of disjoint open boxes, together with their boundaries (which have measure zero, so we can neglect them). Therefore $\int_{\text{open set}} f = 0$. In particular $\int_{\mathbb{R}^n} f = 0$ as well. Thus $\int_{\text{closed set}} f = \int_{\mathbb{R}^n} f - \int_{\text{open set}} f = 0$. From this it follows that $\int_{F} f = 0$ (by expressing $F$ as some suitable almost disjoint countable union of closed sets). $\Box$

Problem Four.

Define $\{ E_k \}_{k=1}^{\infty}$, a family of measurable subsets of $\mathbb{R}^d$, by

$$E_k := \{ x \in [0,1] : \text{the decimal expansion of } x \text{ contains a } 7 \text{ at the } k\text{-th place} \}$$

and note that $m(E_k) = \frac{1}{10}$.

Let

$$E = \{ x \in \mathbb{R}^d : x \in E_k, \text{for infinitely many } k \} = \text{“lim sup”}_{k \to \infty}(E_k).$$

Now, we can write

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k,$$

and therefore $E$ is measurable.

Warning! This construction is a bit hard to think of! (As are all these weird countable unions-and-intersections things.) But it works! Think about it!

Maybe the way to say it is,

$x \in E \iff \text{for all } n \in \mathbb{N}, \text{there exists some } k \geq n \text{ such that } x \in E_k$.”. That will ensure that $x$ is in infinitely many of the $E_k$’s.
Then the giant intersection $\bigcap_{n=1}^{\infty}$ writes down the “for all”, and the giant union $\bigcup_{k\geq n}$ writes down the “there exists”.

Anyway, I claim that $m(E) = 1$. To see this, consider the complement:

$$E^c \cap [0, 1] = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E^c_k \cap [0, 1]$$

$$\Rightarrow m(E^c \cap [0, 1]) = \sum_{n=1}^{\infty} m\left(\bigcap_{k \geq n} E^c_k \cap [0, 1]\right)$$

$$= \sum_{n=1}^{\infty} \inf_{k \geq n} \left(\frac{9}{10}\right)^{k-n}$$

$$= \sum_{n=1}^{\infty} 0$$

$$= 0$$

where we used the fact that, for this particular construction, it’s always the case that $E_j$ and $E_k$ are “independent events” in the sense that $m(E_j \cap E_k) = m(E_j)m(E_k)$ and $m(E^c_j \cap E^c_k) = m(E^c_j)m(E^c_k)$. Note that this is not generally true, but because of the way the $E_k$ have been chosen, it works out this time. □

Remark: Compare this problem with the second Borel-Cantelli lemma, and also the Cantor set construction.