Problem One.
Let $S \subset \mathbb{R}^n$ be a measurable set with $m(S) < \infty$ and let $\epsilon > 0$ be a positive real number. Show that there exists a compact subset $K \subset S$ such that
\[ m(S) - \epsilon \leq m(K) \leq m(S). \]

Proof. If $S$ is measurable, then $m(S) = m^*(S) = \inf_{O \supseteq S} m^*(O)$, so for any $\epsilon > 0$ there exists some open $O_\epsilon \supseteq S$ for which $m(O_\epsilon) - m(S) < \epsilon$.

Now in fact we want to apply this Remark not to $S$ but to $S^c$. Notice that an open set $O_\epsilon \supseteq S^c$ has as its complement a closed set $(O_\epsilon)^c = F_\epsilon \subset S$.

Therefore, for any $\epsilon > 0$ we can find $F_{\epsilon/2} \subset S$ that $m(S) - m(F_{\epsilon/2}) < \frac{\epsilon}{2}$.

Now to generate a compact set $K$, note that a compact set in $\mathbb{R}^n$ is the same as a closed and bounded set. Therefore, for fixed $\epsilon$, let $K_n := B_n(0) \cap F_{\epsilon/2}$, where $B_n(0)$ denotes the closed ball of radius $n$ and centre 0. Now thus defined, $K_n$ is compact for all $n \in \mathbb{N}$,
\[ K_1 \subset K_2 \subset K_3 \subset \cdots \subset F_{\epsilon/2}, \]
and
\[ F_{\epsilon/2} = \bigcup_{n=1}^{\infty} K_n. \]

By monotonicity of measure and by the monotone convergence theorem in $\mathbb{R}$, it follows that
\[ m(F_{\epsilon/2}) = \lim_{n \to \infty} m(K_n) \]
and for any $\epsilon > 0$ there exists $N$ sufficiently large that
\[ m(F_{\epsilon/2}) - \frac{\epsilon}{2} < m(K_N) < m(F_{\epsilon/2}). \]

Therefore,
\[ m(S) - \epsilon < m(K_N) < m(S) \]
as desired. □

Problem Two.
Let $X$ be a set and $\mathcal{M}$ be a $\sigma$–algebra of subsets of $X$. Suppose that $m : \mathcal{M} \to [0, \infty]$ is a finitely additive, countably subadditive function. Show that $m$ is countably additive.

Proof. It suffices to show that for any pairwise disjoint sequence of subsets $S_0, S_1, S_2, \cdots \subset X$, and for any $N \in \mathbb{N}$,
\[ \sum_{k=1}^{N} m(S_k) \leq m(S) \]
where $S = \bigcup_{k=1}^{\infty} S_k$. But indeed, write $S$ as a disjoint union

$$S = \left( \bigcup_{k=1}^{N} S_k \right) \cup \left( \bigcup_{k=N+1}^{\infty} S_k \right)$$

so that by finite additivity,

$$m(S) = m\left( \bigcup_{k=1}^{N} S_k \right) + m\left( \bigcup_{k=N+1}^{\infty} S_k \right) = \sum_{k=1}^{N} m(S_k) + m\left( \bigcup_{k=N+1}^{\infty} S_k \right) \geq \sum_{k=1}^{N} m(S_k)$$

as required. □

Appendix to Problem One.

I typed out this unnecessary lemma before realising it was unnecessary for this problem. Nevertheless, it is interesting and important.

**Lemma.** If $S$ is measurable, then for all $\epsilon > 0$ there exists an open $O_\epsilon \supseteq S$ such that $m^*(O_\epsilon - S) < \epsilon$.

Indeed, if $S$ is measurable, then for any $A \in \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap S) + m^*(A \cap S^c).$$

We also know that $m^*(S) = \inf_{O \supseteq S} m^*(O)$, so for any $\epsilon > 0$ there exists some open $O_\epsilon \supseteq S$ for which $m^*(O_\epsilon) - m^*(S) < \epsilon$. Put $A = O_\epsilon$ into the equation above, to obtain

$$m^*(O_\epsilon) = m^*(O_\epsilon \cap S) + m^*((O_\epsilon)^c \cap S)$$

which is

$$m^*(O_\epsilon) = m^*(S) + m^*((O_\epsilon)^c - S).$$

**Case 1.** Suppose that $m^*(S) = m(S) < \infty$. Then we can subtract it from both sides to get

$$m^*((O_\epsilon)^c - S) = m^*(O_\epsilon) - m^*(S) < \epsilon$$

as required.

**Case 2.** Alternatively, if $m(S) = \infty$, then chop up $S$ into countably many pieces $S_1, S_2, S_3, \ldots$ each of finite measure, and apply Case 1 to each piece, choosing $\epsilon_j = 2^{-j}\epsilon$ for $S_j$. End of Lemma. □

**Remark.** This Lemma furnishes us with a very important alternative definition of what it means to be a measurable set:

**Definition.** $S \subset \mathbb{R}^n$ is *measurable* just if for all $\epsilon > 0$ there exists an open $O_\epsilon \supseteq S$ such that $m^*(O_\epsilon - S) < \epsilon$.

N.B. that I haven’t proved the back direction, namely that this definition entails the prof’s definition of “measurable”. The prof’s definition is usually known as the Caratheodory definition of measurability. The back direction is “left as exercise.” End of Remark.