TABLEAUX COMBINATORICS FOR THE ASYMMETRIC EXCLUSION PROCESS AND ASKEY-WILSON POLYNOMIALS

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Abstract. Introduced in the late 1960's [26, 35], the asymmetric exclusion process (ASEP) is an important model from statistical mechanics which describes a system of interacting particles hopping left and right on a one-dimensional lattice of n sites with open boundaries. It has been cited as a model for traffic flow and protein synthesis. In its most general form, particles may enter and exit at the left with probabilities α and γ, and they may exit and enter at the right with probabilities β and δ. In the bulk, the probability of hopping left is q times the probability of hopping right. The first main result of this paper is a combinatorial formula for the stationary distribution of the ASEP with all parameters general, in terms of a new class of tableaux which we call *staircase tableaux*. This generalizes our previous work [8, 9] for the ASEP with parameters $\gamma = \delta = 0$. Using our first result and also results of Uchiyama-Sasamoto-Wadati [39], we derive our second main result: a combinatorial formula for the moments of Askey-Wilson polynomials. Since the early 1980's there has been a great deal of work giving combinatorial formulas for moments of various other classical orthogonal polynomials (e.g. Hermite, Charlier, Laguerre, Meixner). However, this is the first such formula for the Askey-Wilson polynomials, which are at the top of the hierarchy of classical orthogonal polynomials.

Contents

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1. Introduction

The asymmetric exclusion process (ASEP) is an important model from statistical mechanics which was introduced independently in the context of biology [26] and in mathematics [35] around 1970. Since then there has been a huge amount of activity on the ASEP and its variants for a number of reasons: although the definition of the model is quite simple, the ASEP is surprisingly rich. For example, it exhibits boundary-induced phase transitions, spontaneous symmetry breaking, and phase separation. Furthermore, the ASEP is regarded as a primitive model for translation in protein synthesis [26], traffic flow [32], and formation of shocks [14]; it also appears in a kind of sequence alignment problem in computational biology [5].

There are several common versions of the ASEP: one involves particles hopping on a one-dimensional lattice of n sites with open boundaries; another involves particles hopping on $\mathbb{Z}$. Spectacular progress has been made on the latter model when particles are restricted to hopping in only one direction (this is called the *TASEP* for *totally asymmetric exclusion process*), starting with Johansson's work [21] which reinterprets the TASEP as a randomly growing Young diagram and leads to connections with random matrices. This connection between the TASEP, Young diagrams, and random matrices has been extremely fruitful, see [2, 4, 16, 30]. However, while there has been some recent progress on the more general ASEP on $\mathbb{Z}$ (see for example [3, 38]), a good combinatorial model for it is so far absent from the story.

In this paper we study the ASEP not on $\mathbb{Z}$ but on a one-dimensional lattice of n sites with open boundaries. Particles may enter at the left at rate αdt and exit the system from the right at rate βdt. In the most general form of the model, particles also enter from the right at rate δdt and exit to the left at rate γdt. The probability of hopping left and right is qdt and udt, respectively.[1] An important breakthrough for this model was made by Derrida, Evans, Hakim, and Pasquier [13], who introduced a *matrix ansatz* as a tool for understanding its stationary distribution. More recently, this version of the ASEP has been shown to have remarkable connections to orthogonal polynomials [31, 39], the Bethe ansatz [12, 34], and, under certain specializations of parameters, to combinatorics [15, 7, 36, 8, 9].

The first main result of this paper is a combinatorial formula for the stationary distribution of the ASEP on a finite lattice with all parameters general. Our formula is in terms of some new tableaux we call *staircase tableaux*. These tableaux generalize permutation tableaux (equivalently, alternative tableaux [41]), which were used to describe the stationary distribution of the ASEP when $\gamma = \delta = 0$ [8, 9].

As n goes to infinity, it is natural to expect that the behavior of the ASEP on n sites converges to the behavior of the ASEP on $\mathbb{Z}$. Therefore, one may hope that our new tableaux may be useful in understanding not only the ASEP on a finite lattice, but also the ASEP on $\mathbb{Z}$.

[1] Actually there is no loss of generality in setting $u = 1$, so we will often do so.

The second main result of this paper is a combinatorial formula for the moments of Askey-Wilson polynomials. The Askey-Wilson polynomials are a family of q-orthogonal polynomials with parameters a, b, c, d and q [1]. They reside at the top of the hierarchy of the one-variable q-orthogonal polynomial family in the Askey scheme. In the early 1980's, following groundbreaking work of Flajolet [17], Viennot [40] initiated a combinatorial approach to orthogonal polynomials. Since then, combinatorial formulas have been given for the moments of many of the polynomials in the Askey scheme, including q-Hermite, Tchebycheff, q-Laguerre, Charlier, Meixner, and Al-Salam-Chihara polynomials, see e.g. [19, 20, 24, 23, 27, 33]. However, until now, no such formula was known for the moments of the Askey-Wilson polynomials. To prove our moment formula, we combine our first main result together with results of Uchiyama-Sasamoto-Wadati [39].

It's worth noting that the proof of the stationary distribution with all parameters general is much more difficult than the proof when $\gamma = \delta = 0$ [8, 9]. One interesting feature of our argument is a new generalization of the *matrix ansatz* of Derrida, Evans, Hakim, and Pasquier [13], which is more flexible, albeit harder to use: instead of checking three identities, we must check three infinite families of identites.

We believe that our new tableaux deserve further study, because of their combinatorial interest and their potential connection to geometry. For example, staircase tableaux of size n have cardinality $4^n n!$, and hence are in bijection with doubly-signed permutations. In [10], we will prove this with an explicit bijection, and also develop connections to other combinatorial objects. Furthermore, because of the connection to the ASEP, we know that our staircase tableaux have some hidden symmetries which are not at all apparent from their definition. For instance, it is clear from the definition of the ASEP that the model remains unchanged if we reflect it over the y-axis, and exchange parameters α and δ, β and γ, and q and u. However, the corresponding bijection on the level of tableaux has so far eluded us. Finally, staircase tableaux generalize permutation tableaux, which index certain cells in the non-negative part of the Grassmannian [29]; it would be interesting to better understand the relationship between the tableaux, the ASEP, and the geometry, and potentially generalize it to staircase tableaux.

The structure of this paper is as follows. In Section 2 we define the ASEP, and in Sections 3 and 4 we state our main results on the ASEP and on Askey-Wilson polynomials. In Sections 5, 6 and 7 we prove a generalized matrix ansatz, and our results on the ASEP and Askey-Wilson polynomials. Section 8 gives open problems, and the Appendix describes the relationship between staircase, permutation, and alternative tableaux.

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2. The asymmetric exclusion process (ASEP)

The ASEP is often defined using a continuous time parameter [13]. However, one can equivalently define it as a discrete-time Markov chain [15], as follows.

Definition 2.1. *Let* α, β, γ, δ, q, *and* u *be constants such that* $0 \le \alpha \le 1$, $0 \le \beta \le 1$, $0 \le \gamma \le 1$, $0 \le \delta \le 1$, $0 \le q \le 1$, *and* $0 \le u \le 1$. *Let* B_n *be the set of all* 2^n *words in the language* $\{\circ, \bullet\}^*$. *The ASEP is the Markov chain on* B_n *with transition probabilities:*

- *If* $X = A \bullet \circ B$ *and* $Y = A \circ \bullet B$ *then* $P_{X,Y} = \frac{u}{n+1}$ *(particle hops right) and* $P_{Y,X} = \frac{q}{n+1}$ *(particle hops left).*
- *If* $X = \circ B$ *and* $Y = \bullet B$ *then* $P_{X,Y} = \frac{\alpha}{n+1}$ *(particle enters from left).*
- *If* $X = B\bullet$ *and* $Y = B\circ$ *then* $P_{X,Y} = \frac{\beta}{n+1}$ *(particle exits to the right).*
- *If* $X = \bullet B$ *and* $Y = \circ B$ *then* $P_{X,Y} = \frac{\gamma}{n+1}$ *(particle exits to the left).*
- *If* $X = B\circ$ *and* $Y = B\bullet$ *then* $P_{X,Y} = \frac{\delta}{n+1}$ *(particle enters from the right).*
- *Otherwise* $P_{X,Y} = 0$ *for* $Y \neq X$ *and* $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

Note that we will sometimes denote a state of the ASEP as a word in $\{0, 1\}^n$ and sometimes as a word in $\{\circ, \bullet\}^n$. In the latter notation, the symbol $\circ$ denotes the absence of a particle, which one can also think of as a white particle.

See Figure 1 for an illustration of the four states, with transition probabilities, for the case $n = 2$. The probabilities on the loops are determined by the fact that the sum of the probabilities on all outgoing arrows from a given state must be 1.

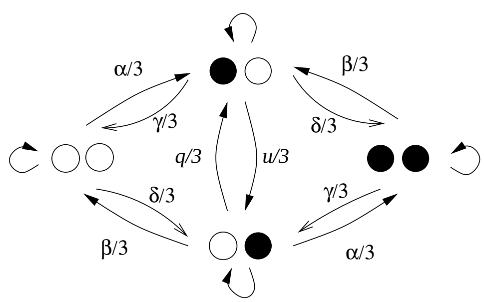

Figure 1. The state diagram of the ASEP for $n = 2$

In the long time limit, the system reaches a steady state where all the probabilities $P_n(\tau_1, \tau_2, \ldots, \tau_n)$ of finding the system in configurations $(\tau_1, \tau_2, \ldots, \tau_n)$ are stationary, i.e. satisfy

$$\frac{d}{dt} P_n(\tau_1, \ldots, \tau_n) = 0.$$

Moreover, the stationary distribution is unique [13], as shown by Derrida *et al.*

The ASEP clearly has multiple symmetries, including the following.

- The "left-right" symmetry: if we reflect the ASEP over the y-axis, we get back the same model, except that the parameters α and δ, γ and β, and u and q are switched.

- The "arrow-reversal" symmetry: if we exchange black and white particles, we get back the same model, except that the parameters α and γ, β and δ, and u and q are switched.
- The "particle-hole" symmetry: if we compose the above two symmetries, i.e. reflect the ASEP over the y-axis and exchange black and white particles, we get back the same model, except that α and β, and γ and δ are switched.

These symmetries imply results about the stationary distribution.

Observation 2.2. *The steady state probabilities satisfy the following identities:*

- $P_n(\tau_1, \dots, \tau_n) = P_n(\tau_n, \dots, \tau_1)|_{\alpha \leftrightarrow \delta, \beta \leftrightarrow \gamma, u \leftrightarrow q}$,
- $P_n(\tau_1, \dots, \tau_n) = P_n(1 - \tau_1, \dots, 1 - \tau_n)|_{\alpha \leftrightarrow \gamma, \beta \leftrightarrow \delta, u \leftrightarrow q}$,
- $P_n(\tau_1, \dots, \tau_n) = P_n(1 - \tau_n, \dots, 1 - \tau_1)|_{\alpha \leftrightarrow \beta, \gamma \leftrightarrow \delta}$.

Above, the notation $|_{\alpha \leftrightarrow \delta}$ indicates that the parameters α and δ are exchanged.

3. Staircase tableaux and the stationary distribution of the ASEP

The main combinatorial objects of this paper are some new tableaux which we call *staircase tableaux*. These tableaux generalize permutation tableaux (equivalently, alternative tableaux).

Definition 3.1. *A* staircase tableau *of size n is a Young diagram of "staircase" shape* $(n, n-1, \dots, 2, 1)$ *such that boxes are either empty or labeled with α, β, γ, or δ, subject to the following conditions:*

- *no box along the diagonal is empty;*
- *all boxes in the same row and to the left of a β or a δ are empty;*
- *all boxes in the same column and above an α or a γ are empty.*

The type *of a staircase tableau is a word in $\{\circ, \bullet\}^n$ obtained by reading the diagonal boxes from northeast to southwest and writing a $\bullet$ for each α or δ, and a $\circ$ for each β or γ.*

See the left of Figure 2 for an example of a staircase tableau.

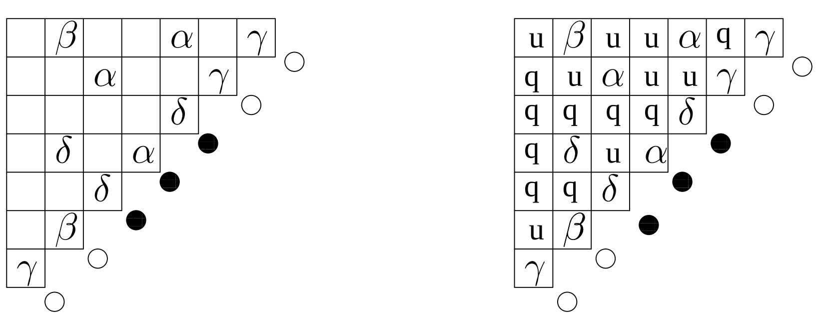

Figure 2. A staircase tableau of size 7 and type $\circ\circ\bullet\bullet\bullet\circ\circ$

Definition 3.2. *The* weight $\mathrm{wt}(\mathcal{T})$ *of a staircase tableau* $\mathcal{T}$ *is a monomial in* $\alpha, \beta, \gamma, \delta, q$, *and* u, *which we obtain as follows. Every blank box of* $\mathcal{T}$ *is assigned a* q *or* u, *based on the label of the closest labeled box to its right in the same row and the label of the closest labeled box below it in the same column, such that:*

- *every blank box which sees a* β *to its right gets assigned a* u;
- *every blank box which sees a* δ *to its right gets assigned a* q;
- *every blank box which sees an* α *or* γ *to its right, and an* α *or* δ *below it, gets assigned a* u;
- *every blank box which sees an* α *or* γ *to its right, and a* β *or* γ *below it, gets assigned a* q.

After assigning a q *or* u *to each blank box in this way, the* weight *of* $\mathcal{T}$ *is then defined as the product of all labels in all boxes.*

The right of Figure 2 shows that this staircase tableau has weight $\alpha^3\beta^2\gamma^3\delta^3q^9u^8$.

Remark 3.3. *The weight of a staircase tableau always has degree* $n(n+1)/2$. *For convenience, we will sometimes set* $u = 1$, *since this results in no loss of information.*

Our first main result (to be proved in Section 6) is the following.

Theorem 3.4. *Consider any state* τ *of the ASEP with* n *sites, where the parameters* $\alpha, \beta, \gamma, \delta, q$ *and* u *are general. Set* $Z_n = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, *where the sum is over all staircase tableaux of size* n. *Then* Z_n *is the partition function for the ASEP, and the steady state probability that the ASEP is at state* τ *is precisely*

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_n},$$

where the sum is over all staircase tableaux $\mathcal{T}$ *of type* τ.

Figure 3 illustrates Theorem 3.4 for the state ●● of the ASEP. All staircase tableaux $\mathcal{T}$ of type ●● are shown. It follows that the steady state probability of ●● is

$$\frac{\alpha^2u + \delta^2q + \alpha\delta q + \alpha\delta u + \alpha^2\delta + \alpha\beta\delta + \alpha\gamma\delta + \alpha\delta^2}{Z_2}.$$

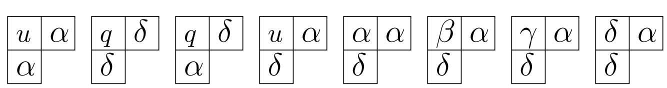

FIGURE 3. The tableaux of type ●●

We can also obtain combinatorial formulas for various physical quantities. Theorem 3.5 below will be proved in Section 6.4.

Theorem 3.5. *Consider the ASEP with* n *sites. Then we have the following:*

- *The current in the steady state is* $\frac{Z_{n-1}}{Z_n}$, *where* Z_n *is the generating function for staircase tableaux of size* n.

- *The average particle number* $\langle \tau_i \rangle_n$ *at site* i *is given by* Z_n^{-1} *times the generating function for all staircase tableaux of size* n *which have an* α *or* δ *at the* i*th position along the diagonal.*
- *Similarly, the* m*-point function* $\langle \tau_{i_1} \dots \tau_{i_m} \rangle_n$ *is given by* Z_n^{-1} *times the generating function for all staircase tableaux of size* n *which have an* α *or* δ *in positions* $i_1, i_2, \dots,$ *and* i_m *along the diagonal.*

Remark 3.6. *In* [8] *and* [9]*, which concerned the ASEP with parameters* $\gamma = \delta = 0$*, we gave combinatorial expressions for the stationary distribution in terms of permutation tableaux: more specifically, the steady state probability of a state* τ *was given by enumerating permutation tableaux lying in a Young diagram of shape* $\lambda(\tau)$*, according to the number of* 1*'s in the top row and the number of unrestricted rows.*

Subsequently [6] *and* [11] *observed that a permutation tableau is determined by complementary statistics, namely the positions of the topmost* 1*'s not in the first row, and rightmost restricted zero's. Then* [41] *took this observation a step further, defining some equivalent* alternative tableaux*, by making boxes red, blue, or empty (corresponding to whether the box of the associated permutation tableau contained a topmost* 1*, rightmost restricted zero, or neither). Our staircase tableaux generalize both kinds of tableaux, but now have non-empty boxes labeled by* $\alpha, \beta, \gamma, \delta$*. Additionally, instead of working with Young diagrams of various shapes, we now always work with a staircase shape, whose* type *encodes the same information (namely, the corresponding state of the ASEP) that the shape used to. Working with this larger shape seems to be the only natural way to assign the appropriate powers of* q *and* u *to the tableau.*

Remark 3.7. *As an alternative to Definition 3.2, suppose we define the* dual weight $\mathrm{wt}'(\mathcal{T})$ *of a staircase tableau by taking the product of all labels in all boxes, after having filled the blank boxes of* $\mathcal{T}$ *according to the following rule:*

- *every blank box which sees an* α *below it gets assigned a* u*;*
- *every blank box which sees a* γ *below it gets assigned a* q*;*
- *every blank box which sees an* α *or* δ *to its right, and a* β *or* δ *below it, gets assigned a* q*;*
- *every blank box which sees a* β *or* γ *to its right, and a* β *or* δ *below it, gets assigned a* u*.*

Then Theorem 3.4 continues to hold, with wt *replaced by* wt$'$*. This follows from the left-right symmetry of the ASEP. More specifically, note that if* $\mathcal{T}$ *is a staircase tableau, then the tableau* $\mathcal{T}'$ *obtained by transposing* $\mathcal{T}$ *then switching* α *and* δ*, and* β *and* γ *is still a staircase tableau. Our observation now follows from Theorem 3.4, the fact that* $\mathrm{wt}'(\mathcal{T}') = \mathrm{wt}(\mathcal{T})$*, and Observation 2.2. (Alternatively, we could have proved the observation using a method analogous to the proof of Theorem 3.4.)*

4. Askey-Wilson polynomials and a formula for their moments

The Askey-Wilson polynomial is a q-orthogonal polynomial with four free parameters besides q. It resides at the top of the hierarchy of the one-variable q-orthogonal polynomial family in the Askey scheme [1, 18, 25]. In this section we define the Askey-Wilson polynomials, following the exposition of [1] and [39], then state a combinatorial formula for their moments.

The q-shifted factorial is defined by

$$(a_1, a_2, \cdots, a_s; q)_n = \prod_{r=1}^{s} \prod_{k=0}^{n-1} (1 - a_r q^k),$$

and the basic hypergeometric function is given by

$${}_r\phi_s \left[\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(b_1, \cdots, b_s, q; q)_k} ((-1)^k q^{k(k-1)/2})^{1+s-r} z^k.$$

The Askey-Wilson polynomial $P_n(x) = P_n(x; a, b, c, d|q)$ is explicitly defined by

$$P_n(x) = a^{-n}(ab, ac, ad; q)_n \; {}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} ; q, q \right],$$

with $x = \cos\theta$ for $n \in \mathbb{Z}_+ := \{0, 1, 2, \cdots\}$. It satisfies the three-term recurrence

$$A_n P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x) = 2x P_n(x),$$

with $P_0(x) = 1$ and $P_{-1}(x) = 0$, where

$$A_n = \frac{1 - q^{n-1}abcd}{(1 - q^{2n-1}abcd)(1 - q^{2n}abcd)},$$

$$B_n = \frac{q^{n-1}}{(1 - q^{2n-2}abcd)(1 - q^{2n}abcd)} [(1 + q^{2n-1}abcd)(qs + abcds') - q^{n-1}(1 + q)abcd(s + qs')],$$

$$C_n = \frac{(1 - q^n)(1 - q^{n-1}ab)(1 - q^{n-1}ac)(1 - q^{n-1}ad)(1 - q^{n-1}bc)(1 - q^{n-1}bd)(1 - q^{n-1}cd)}{(1 - q^{2n-2}abcd)(1 - q^{2n-1}abcd)},$$

$$\text{and} \quad s = a + b + c + d, \qquad s' = a^{-1} + b^{-1} + c^{-1} + d^{-1}.$$

For $|a|, |b|, |c|, |d| < 1$, using $z = e^{i\theta}$, the orthogonality is expressed by

$$\oint_C \frac{dz}{4\pi i z} w\left(\frac{z + z^{-1}}{2}\right) P_m\left(\frac{z + z^{-1}}{2}\right) P_n\left(\frac{z + z^{-1}}{2}\right) = h_n \delta_{mn},$$

where the integral contour C is a closed path which encloses the poles at $z = aq^k$, bq^k, cq^k, dq^k ($k \in \mathbb{Z}_+$) and excludes the poles at $z = (aq^k)^{-1}$, $(bq^k)^{-1}$, $(cq^k)^{-1}$, $(dq^k)^{-1}$

($k \in \mathbb{Z}_+$), and where

$$w(\cos\theta) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty},$$

$$\frac{h_n}{h_0} = \frac{(1 - q^{n-1}abcd)(q, ab, ac, ad, bc, bd, cd; q)_n}{(1 - q^{2n-1}abcd)(abcd; q)_n},$$

$$h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}.$$

(In the other parameter region, the orthogonality is continued analytically.)

The moments are defined by

$$\mu_k = \oint_C \frac{dz}{4\pi i z} w\left(\frac{z + z^{-1}}{2}\right)\left(\frac{z + z^{-1}}{2}\right)^k.$$

The second main result of this paper is a combinatorial formula for the moments of the Askey-Wilson polynomials. In Theorem 4.1 below, we use the substitution

$$\alpha = \frac{1-q}{1 + ac + a + c}, \qquad \beta = \frac{1-q}{1 + bd + b + d},$$

$$\gamma = \frac{-(1-q)ac}{1 + ac + a + c}, \qquad \delta = \frac{-(1-q)bd}{1 + bd + b + d},$$

which can be inverted via

$$\begin{aligned} a &= \frac{1 - q - \alpha + \gamma + \sqrt{(1 - q - \alpha + \gamma)^2 + 4\alpha\gamma}}{2\alpha} \\ c &= \frac{1 - q - \alpha + \gamma - \sqrt{(1 - q - \alpha + \gamma)^2 + 4\alpha\gamma}}{2\alpha} \\ b &= \frac{1 - q - \beta + \delta + \sqrt{(1 - q - \beta + \delta)^2 + 4\beta\delta}}{2\beta} \\ d &= \frac{1 - q - \beta + \delta - \sqrt{(1 - q - \beta + \delta)^2 + 4\beta\delta}}{2\beta}. \end{aligned}$$

Recall that $Z_\ell = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, where the sum is over all staircase tableaux of size ℓ.

Theorem 4.1. *The kth moment of the Askey-Wilson polynomials is given by*

$$\mu_k = h_0 \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \left(\frac{1-q}{2}\right)^\ell \frac{Z_\ell}{\prod_{i=0}^{\ell-1}(\alpha\beta - \gamma\delta q^i)}.$$

5. A more flexible matrix ansatz

One of the most powerful techniques for studying the ASEP is the so-called *matrix ansatz*, an ansatz given by Derrida, Evans, Hakim, and Pasquier [13] as a tool for solving for the steady state probabilities $P_n(\tau_1, \ldots, \tau_n)$ of the ASEP. In this section we will start by recalling their matrix ansatz, and then give a new generalization of it which we require for our proof of Theorem 3.4.

For convenience, in this section we set $u = 1$. Also, we define unnormalized weights $f_n(\tau_1, \dots, \tau_n)$, which are equal to the $P_n(\tau_1, \dots, \tau_n)$ up to a constant:

$$P_n(\tau_1, \dots, \tau_n) = f_n(\tau_1, \dots, \tau_n)/Z_n,$$

where Z_n is the *partition function* $\sum_\tau f_n(\tau_1, \dots, \tau_n)$. The sum defining Z_n is over all possible configurations $\tau \in \{0,1\}^n$. Derrida *et al* showed the following.

Theorem 5.1. [13] *Suppose that D and E are matrices, V is a column vector, and W is a row vector, such that the following conditions hold:*

$$DE - qED = D + E, \ \beta DV - \delta EV = V, \ \alpha WE - \gamma WD = W.$$

Then for any state $\tau = (\tau_1, \dots, \tau_n)$ of the ASEP,

$$f_n(\tau_1, \dots, \tau_n) = W(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))V.$$

Note that $\prod_{i=1}^n(\tau_i D + (1-\tau_i)E)$ is simply a product of n matrices D or E with matrix D at position i if site i is occupied ($\tau_i = 1$). Also note that Theorem 5.1 implies that $Z_n = W(D+E)^n V$.

We now state and prove a more flexible version of Theorem 5.1. Our proof generalizes the argument given in [13].

Theorem 5.2. *Let $\{\lambda_n\}_{n\geq 0}$ be a family of constants. Let W and V be row and column vectors, and D and E be matrices such that for any words X and Y in D and E, we have:*

(I) $WX(DE - qED)YV = \lambda_{|X|+|Y|+2} WX(D+E)YV$;
(II) $\beta WXDV - \delta WXEV = \lambda_{|X|+1} WXV$;
(III) $\alpha WEYV - \gamma WDYV = \lambda_{|Y|+1} WYV$.

(Here $|X|$ is the length of X.) Then for any state $\tau = (\tau_1, \dots, \tau_n)$ of the ASEP,

$$f_n(\tau) = W(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))V.$$

Proof. We are in the steady state of the ASEP if the net rate of entering each state $(\tau_1, \dots, \tau_n)$ is 0, or in other words, the following expression equals 0:

$$(1) \qquad (-1)^{\tau_1}(-\alpha f_n(0, \tau_2, \dots, \tau_n) + \gamma f_n(1, \tau_2, \dots, \tau_n))$$

$$(2) \qquad + \sum_{i=1}^{n-1}(-1)^{\tau_i}\chi(\tau_i \neq \tau_{i+1})(f_n(\tau_1, \dots, 1, 0, \dots, \tau_n) - q f_n(\tau_1, \dots, 0, 1, \dots, \tau_n))$$

$$(3) \qquad +(-1)^{\tau_n}(\beta f_n(\tau_1, \dots, \tau_{n-1}, 1) - \delta f_n(\tau_1, \dots, \tau_{n-1}, 0)).$$

In (2) above, the arguments $1, 0$ and $0, 1$ are in positions i and $i+1$, and χ is the boolean function taking value 1 or 0 based on whether its argument is true or false.

Now suppose there is a family of constants x_1^n, x_0^n (for each positive n) such that:

- the expression (1) above equals $x_{\tau_1}^n f_{n-1}(\tau_2, \dots, \tau_n)$,
- (2) equals $\sum_{i=1}^n(-x_{\tau_i}^n f_{n-1}(\tau_1, \dots, \hat{\tau}_i, \dots, \tau_n) + x_{\tau_{i+1}}^n f_{n-1}(\tau_1, \dots, \hat{\tau}_{i+1}, \dots, \tau_n))$,

- and (3) equals $-x^n_{\tau_n} f_{n-1}(\tau_1, \ldots, \tau_{n-1})$.

(Here $\hat{\tau}_i$ denotes the omission of the ith component.) Then it follows from these conditions that the sum of (1), (2), and (3) is equal to 0 (and hence we are in the steady state), since all terms involving f_{n-1} cancel out.

Finally, note that the constants $x^n_1 = \lambda_n$ and $x^n_0 = -\lambda_n$ together with (I), (II), and (III) of Theorem 5.2 satisfy these conditions.

□

6. The proof of the stationary distribution

In this section we will prove Theorem 3.4 by: defining vectors W, V and matrices D, E; proving that they have the requisite combinatorial interpretation in terms of staircase tableaux; and checking that they satisfy the relations of Theorem 5.2, with $\lambda_0 = 1$ and $\lambda_n = \alpha\beta - \gamma\delta q^{n-1}$ for $n \geq 1$.

This is analogous to the proof of [8, Theorem 3.1], albeit much more difficult: in [8], it was obvious that our matrices and vectors satisfied the matrix ansatz [8, Lemma 2.5], and easy to show that our combinatorial objects were described by the algebraic relations of the ansatz, see [8, Figure 6] and the surrounding discussion.

In contrast, in this more general situation, we can give a combinatorial proof of relation (III) of our new matrix ansatz, but not for (I) or (II). Instead we give a rather difficult algebraic proof of (I) and (II). First of all, our new "vectors" and "matrices" have two and four indices, respectively, which makes working with them more complicated. Second, to use Theorem 5.2, instead of proving that our vectors and matrices satisfy three identities (as in Theorem 5.1), we must prove that they satisfy three *infinite families* of identities. Moreover, there is no obvious way to use induction to prove these identities: one cannot take one of the identities and multiply on the left or right to obtain the next identity in the family.

In this section we assume $u = 1$. Recall that this is no loss of generality, as the weight of a staircase tableau of size n is always a monomial of degree $n(n+1)/2$. However, setting $t = 1$ has the advantage of making our bookkeeping a little easier.

6.1. The definition of our matrices.

Definition 6.1. *We define "row" and "column" vectors $W = (W_{ik})_{i,k}$ and $V = (V_{j\ell})_{j,\ell}$, and matrices $D = (D_{i,j,k,\ell})_{i,j,k,\ell}$ and $E = (E_{i,j,k,\ell})_{i,j,k,\ell}$ (where i, j, k, l are all non-negative) by the following:*

$$W_{ik} = \begin{cases} 1 & \text{if } i = k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$V_{j\ell} = 1 \textit{ always.}$$

$$D_{ijk\ell} = \begin{cases} 0 & \text{if } j < i \text{ or } \ell > k+1, \\ \delta q^i & \text{if } i = j-1 \text{ and } k = \ell = 0, \\ \alpha q^i & \text{if } i = j,\ k = 0 \text{ and } \ell = 1, \\ \delta(D_{i,j-1,k-1,\ell} + E_{i,j-1,k-1,\ell}) + D_{i,j,k-1,\ell-1} & \text{otherwise.} \end{cases}$$

$$E_{ijk\ell} = \begin{cases} 0 & \text{if } j < i \text{ or } \ell > k+1, \\ \beta q^i & \text{if } i = j \text{ and } k = \ell = 0, \\ \gamma q^i & \text{if } i = j,\ k = 0 \text{ and } \ell = 1, \\ \beta(D_{i,j,k-1,\ell} + E_{i,j,k-1,\ell}) + qE_{i,j,k-1,\ell-1} & \text{otherwise.} \end{cases}$$

6.2. **The combinatorial interpretation of our matrices in terms of tableaux.** We say that a row of a staircase tableau $\mathcal{T}$ is *indexed by* β if the leftmost box in that row which is not occupied by a q or u is a β. Note that every box to the left of that β must be a u. Similarly we will talk about rows which are *indexed by* δ; in this case, every box to the left of that δ must be a q. We will also talk about rows which are *indexed by* α/γ, which is shorthand for rows which are indexed by α *or* γ.

Theorem 6.2. *If X is a word in D's and E's, then:*

- *$X_{ijk\ell}$ is the generating function for all ways of adding $|X|$ new columns to a staircase tableau with i rows indexed by δ and k rows indexed by α/γ, so as to obtain a new tableau with j rows indexed by δ and ℓ rows indexed by α/γ.*
- *$(WX)_{j\ell}$ is the generating function for staircase tableaux of type X which have j rows indexed by δ and ℓ rows indexed by α/γ (and hence $|X| - j - \ell$ rows indexed by β.)*
- *WXV is the generating function for all staircase tableaux of type X.*

The main step in proving Theorem 6.2 is the following lemma.

Lemma 6.3. *$D_{i,j,k,\ell}$ is the generating function for the weights of all possible new columns with an α or δ in the bottom box that we could add to the left of a staircase tableau with i columns indexed by δ and k columns indexed by α or γ, obtaining a new staircase tableau which has j columns indexed by δ and ℓ columns indexed by α or γ. Similarly for $E_{i,j,k,\ell}$, where the new column has a β or γ in the bottom box.*

Proof. Let $D'_{ijk\ell}$ denote the generating function for all possible new columns with an α or δ in the bottom box that we could add to the left of a staircase tableau with i columns indexed by δ and k columns indexed by α/γ, obtaining a new staircase tableau which has j columns indexed by δ and ℓ columns indexed by α or γ. We will show that $D'_{ijk\ell} = D_{ijk\ell}$ by showing that D' satisfies the same recurrences.

Note that $D'_{ijk\ell} = 0$ if $j < i$ because adding a new column to a staircase tableau never decreases the number of rows indexed by δ. Also $D'_{ijk\ell} = 0$ if $\ell > k+1$ because when we add a new column we can never increase the number of rows indexed by α/γ by more than 1.

Now suppose that $k = 0$. If we are starting from a tableau with i rows indexed by δ and 0 rows indexed by α/γ, then the only way to add a new column is to add a column with an α or δ at the bottom, with all boxes above empty. If we add a δ, then the resulting tableau has $\ell = 0$ rows indexed by α/γ and $j = i+1$ rows indexed by δ. The weight of the new column will be δq^i. On the other hand, if we add an α at the bottom, then the resulting tableau has $\ell = 1$ rows indexed by α/γ and $j = i$ rows indexed by δ. The weight of the new column will be αq^i. From this discussion it follows that $D'_{ijk\ell} = \delta q^i$ when $j = i+1$ and $k = \ell = 0$, and $D'_{ijk\ell} = \alpha q^i$ when $j = i$, $k = 0$, and $\ell = 1$.

In all other situations, we can assume that $k \geq 1$. Suppose that we are adding a new column C with an α or δ at the bottom to the left of a staircase tableau with i columns indexed by δ and k columns indexed by α/γ. Consider the lowest box B of C which is indexed by an α or γ (such a box exists since $k \geq 1$). If we fill B with an α, β, γ or δ, then the bottom box of C must contain a δ. In this case, if we ignore that bottom δ, then our choices for C are exactly the same as our choices would be for adding a new column to the left of a staircase tableau with i columns indexed by δ and $k-1$ columns indexed by α/γ. Therefore, filling B with an α, β, γ or δ gives us a contribution of $d(D'+E')_{i,j-1,k-1,\ell}$ to our generating function.

On the other hand, if we leave B empty, then this box will get a weight t. Filling the rest of the column C is like adding a new column to a staircase tableau with i rows indexed by δ and $k-1$ rows indexed by α/γ. Therefore leaving B empty gives us a contribution of $D'_{i,j,k-1,\ell-1}$ to our generating function.

It follows that when $k \geq 1$, $D'_{ijk\ell} = \delta(D'+E')_{i,j-1,k-1,\ell} + D'_{i,j,k-1,\ell-1}$.

Similarly, we define $E'_{ijk\ell}$ to be the generating function for all possible new columns with a β or γ in the bottom box that we could add to the left of a staircase tableau with i columns indexed by δ and k columns indexed by α/γ, obtaining a new staircase tableau which has j columns indexed by δ and ℓ columns indexed by α/γ. The proof that $E'_{ijk\ell} = E_{ijk\ell}$ is analogous to the proof we gave for D'. □

We now turn to the proof of Theorem 6.2.

Proof of Theorem 6.2. The first item follows from Lemma 6.3 and the definition of matrix multiplication. Multiplying at the left by a W has the effect that we start with the empty tableau and then add columns according to X: so $WX_{j\ell}$ is the generating function for staircase tableaux of type X which have j rows indexed by δ and ℓ rows indexed by α/γ. Finally, multiplying $WX_{j\ell}$ on the right by V has the effect of summing over all δ and ℓ, so WXV is the generating function for all staircase tableaux of type X. □

6.3. **The proof that our matrices satisfy the matrix ansatz.** We now need to prove that our matrices satisfy relations (I.), (II.), and (III.) of Theorem 5.2. Relation (III.) has a simple combinatorial proof. However, this proof does *not* work for relation (II.), and indeed it will require a lot more work to prove (I.) and (II.).

Lemma 6.4. *Relation (III.) of Theorem 5.2 holds.*

Proof. Using Theorem 6.2, relation (III.) can be reformulated in terms of staircase tableaux. First we rewrite (III.) as

$$\alpha WEYV + \gamma\delta q^{n-1}WYV = \gamma WDYV + \alpha\beta WYV,$$

where $n-1 = |Y|$. Since a "type E" corner box of a staircase tableau must be either a β or γ, we can rewrite this again as

$$\alpha WE_\beta YV + \alpha WE_\gamma YV + \gamma\delta q^{n-1}WYV = \gamma WD_\alpha YV + \gamma WD_\delta YV + \alpha\beta WYV.$$

Here $WE_\beta YV$ denotes the generating function for staircase tableaux of type EY, whose northeast corner box is a β; the terms $WE_\gamma YV$, $WD_\alpha YV$, and $WD_\delta YV$ are defined analogously.[2]

It is now easy to see that $\alpha WE_\beta YV = \alpha\beta WYV$ and $\gamma WD_\alpha YV = \gamma\delta q^{n-1}WYV$, since a box labeled β must have only empty boxes (weighted $u=1$) to its left, and a box labeled δ must have only empty boxes (weighted q) to its left. Also, since the rules for the weight of an empty box which sees a γ to its right are the same as the rules for the weight of an empty box which sees an α to its right, we have that $\alpha WE_\gamma YV = \gamma WD_\alpha YV$. This proves relation (III.). □

Lemma 6.5. *For any word Y in D and E, we have that $Y_{ijk\ell} = q^{|Y|}Y_{i-1,j-1,k,\ell}$.*

Proof. We use Theorem 6.2. Note that both $Y_{ijk\ell}$ and $Y_{i-1,j-1,k,\ell}$ enumerate the ways of adding $|Y|$ new columns to a staircase tableau $\mathcal{T}$ so as to increase by $j-i$ the number of rows indexed by δ, and to increase by $\ell-k$ the number of rows indexed by α/γ. The only difference is the initial number of rows indexed by δ. Since $Y_{ijk\ell}$ has one extra initial row indexed by δ, this will contribute $|Y|$ extra empty boxes which all get the weight q. Therefore $Y_{ijk\ell} = q^{|Y|}Y_{i-1,j-1,k,\ell}$. □

Proposition 6.6. *To prove (I.) and (II.), it suffices to prove the following identities for all non-negative integers j and ℓ:*

(1) $(WXDE)_{j\ell} = q(WXED)_{j\ell} + \alpha\beta(WX(D+E))_{j\ell} - \gamma\delta q^{|X|+1}(WX(D+E))_{j-1,\ell}$.
(2) $\beta(WXD)_{j\ell} = \delta(WXE)_{j-1,\ell} + \alpha\beta(WX)_{j,\ell-1} - \gamma\delta q^{|X|}(WX)_{j-1,\ell-1}$.

Proof. We claim the following: if (1) is true, then for any word Y in D's and E's, $(WXDEY)_{j\ell}$ is equal to

$$q(WXEDY)_{j\ell} + \alpha\beta(WX(D+E)Y)_{j\ell} - \gamma\delta q^{|X|+|Y|+1}(WX(D+E)Y)_{j-1,\ell}.$$

To prove the claim, let Y be any word in D and E. Then $(WXDEY)_{j\ell}$ is equal to:

$$\begin{aligned} & \sum_{i,k}(WXDE)_{ik}Y_{ijk\ell} \\ = \; & \sum_{i,k} q(WXED)_{ik}Y_{ijk\ell} + \alpha\beta(WX(D+E))_{ik}Y_{ijk\ell} - \gamma\delta q^{|X|+1}\sum_{ik}(WX(D+E))_{i-1,k}Y_{ijkl} \\ = \; & q(WXEDY)_{j\ell} + \alpha\beta(WX(D+E)Y)_{j\ell}\gamma\delta q^{|X|+|Y|+1}(WX(D+E)Y)_{j-1,\ell}. \end{aligned}$$

[2]We could have defined matrices $D_\alpha, D_\delta, E_\beta, E_\gamma$ in such a way that they have this combinatorial interpretation; and then defined $D = D_\alpha + D_\delta$ and $E = E_\beta + E_\gamma$.

To deduce the final equality above, we applied Lemma 6.5 to the last term.

Now note that if we take the equation of the claim, and sum over all j and ℓ, then we get precisely (I.) (since multiplication on the right by V has the effect of summing over all indices). And if we take (2.) and sum over all j and ℓ, we get (II.) This completes the proof. □

By Proposition 6.6, to prove Theorem 3.4, it is enough to prove the following.

Theorem 6.7. *The following identities hold for all non-negative j and ℓ.*

(1) $(WXDE)_{j\ell} = q(WXED)_{j\ell}+\alpha\beta(WX(D+E))_{j\ell}-\gamma\delta q^{|X|+1}(WX(D+E))_{j-1,\ell}$.

(2) $\beta(WXD)_{j\ell} = \delta(WXE)_{j-1,\ell} + \alpha\beta(WX)_{j,\ell-1} - \gamma\delta q^{|X|}(WX)_{j-1,\ell-1}$.

Note that every term in (1) above is the product of $\delta^j\beta^{|X|+2-j-\ell}$ and a polynomial in variables α, γ and q only. This is easy to see from the combinatorial interpretation of the terms: for example, $(WXDE)_{j\ell}$ is the generating function for certain staircase tableaux of size $|X|+2$, with j rows indexed by δ and ℓ rows indexed by α/γ (which implies that $|X|+2-j-\ell$ are indexed by β. Every term in (2) is also the product of $\delta^j\beta^{|X|+2-j-\ell}$ and a polynomial in variables α, γ and q only.

We will prove Theorem 6.7 by proving both identities simultaneously by induction on the degree in β and δ in these identities, and also on the length of the word in D and E (equivalently, the size of the corresponding tableaux). Therefore it will be convenient for us to make the following notation.

Definition 6.8. *We refer to identity (1) in Theorem 6.7 as $1^n_{j,m}$, where $n = |X|+2$ and $m = |X|+2-j-\ell$. And we will refer to identity (2) above as $2^n_{j,m}$ where $n = |X|+1$ and $m = |X|+2-j-\ell$. For identity (2), if $X = \tilde{X}D$ (where $\tilde{X}$ is some word in D and E), we will refer to the identity as $(2D)$ or $2D^n_{j,m}$. Similarly if $X = \tilde{X}E$.*

Our plan for proving Theorem 6.7 is to demonstrate the following implications:

Proposition 6.9. *The following hold for all j, m, and n.*

$$2^{n-1}_{j',m'}(\text{for } j' \leq j, m' \leq m) \text{ and } 1^n_{j-1,m} \Rightarrow 2D^n_{j,m}.$$
$$2^{n-1}_{j',m'}(\text{for } j' \leq j, m' \leq m) \text{ and } 1^n_{j,m-1} \Rightarrow 2E^n_{j,m}.$$
$$2^{n-1}_{j',m'}(\text{for } j' \leq j, m' \leq m) \text{ and } 2E^n_{j,m} \Rightarrow 1^n_{j,m}.$$

Note that after replacing $2E^n_{j,m}$ in the third implication by $2^{n-1}_{j',m'}$(for $j' \leq j, m' \leq m$) and $1^n_{j,m-1}$ (using the second implication), it is clear that Proposition 6.9 enables us to prove identities (1) and (2) in Theorem 6.7 by induction on the sum $m+j+n$. (Of course we should also check suitable base cases.)

Lemma 6.10. *Theorem 6.7 holds when $n \leq 1$ (and j and m arbitrary) and when $j = 0$ (with n and m arbitrary) and when $m = 0$ (with n and j arbitrary).*

Proof. When $n = 0$ there is nothing to prove. When $n = 1$, (1) is vacuous and (2) can be proved by inspection. There are only four staircase tableaux of size 1 (single boxes filled with α, β, γ, or δ), and it is immediate that: $\beta WD_{0,1} = \alpha\beta W_{0,0} = \alpha\beta$, $\beta D_{1,0} = \delta E_{0,0} = \beta\delta$, and $\gamma\delta W_{0,0} = \delta E_{0,1}$. For all other values of j and ℓ, all terms of equation (2) are 0.

Now suppose that $j = 0$. First consider (1). We must show that $(WXDE)_{j\ell} = q(WXED)_{j\ell} + \alpha\beta(WX(D+E))_{j\ell}$, and since $j = 0$, we should interpret each term as a generating function for tableaux which contain no δ's. We will prove this by exhibiting a bijection for the tableaux corresponding to each term. The term $(WXDE)_{j\ell}$ describes tableaux whose lowest diagonal box is a β or γ, with an α in the adjacent diagonal box. Consider such a tableau $\mathcal{T}$: if the box just above the β or γ is empty, then we map $\mathcal{T}$ to the tableau obtained by swapping the entries of the lowest two diagonal boxes. The weight of the new tableau is q^{-1} times the weight of $\mathcal{T}$. Otherwise, if the box just above the β or γ contains an α, β or γ, then the lowest diagonal box must be a β, and if we delete the lowest diagonal box and the columns second from the left (which contains an α with empty boxes above, all weighted by $u = 1$), then we obtain a tableau corresponding to the set $\alpha\beta(WX(D+E))_{j\ell}$. See Figure 4. It is easy to show that the maps we've described are bijections.

Now consider $j = 0$ and (2). We must show that $(WXD)_{j\ell} = \alpha(WX)_{j,\ell-1}$. This is immediate, as the leftmost column of a tableau from the set $(WXD)_{j\ell}$ must contain an α with empty boxes above, all weighted by $u = 1$. Deleting the column gives rise to a tableau from the set corresponding to $\alpha(WX)_{j,\ell-1}$.

The proofs of (1) and (2) when $m = 0$ are analogous. □

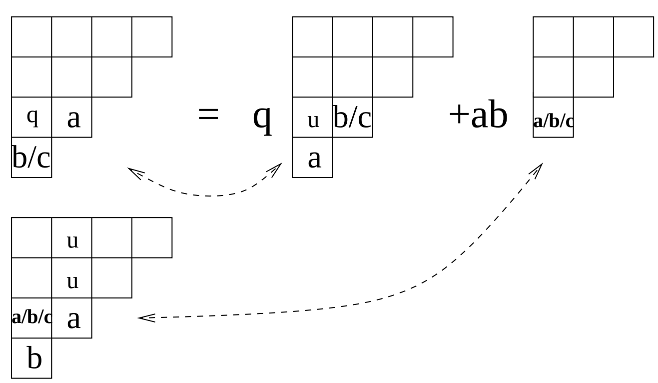

FIGURE 4

We now prove Proposition 6.9.

Proof of Proposition 6.9. We start by demonstrating the first implication which proves $2D^n_{j,m}$, so we have $X = \tilde{X}D$, $n = |X| + 1$, and $m = |X| + 2 - j - \ell$. Note that

$\beta(WXD)_{j\ell} = \beta(W\tilde{X}DD)_{j\ell} = \beta\sum_{i,k}(W\tilde{X}D)_{ik}D_{ijk\ell}$ is equal to:

$$\begin{aligned}
&\sum_{i,k}\delta(W\tilde{X}E)_{i-1,k}D_{ijk\ell} + \sum_{i,k}\alpha\beta(W\tilde{X})_{i,k-1}D_{ijk\ell} - \gamma\delta q^{|\tilde{X}|}\sum_{i,k}(W\tilde{X})_{i-1,k-1}D_{ijk\ell} \\
= \; & q\delta(W\tilde{X}ED)_{j-1,\ell} + \sum_{i,k}\alpha\beta(W\tilde{X})_{i,k-1}(\delta(D+E)_{i,j-1,k-1,\ell} + D_{i,j,k-1,\ell-1}) \\
& -\gamma\delta q^{|\tilde{X}|}\sum_{i,k}(W\tilde{X})_{i-1,k-1}(q\delta(D+E)_{i-1,j-2,k-1,\ell} + D_{i-1,j-1,k-1,\ell-1}) \\
= \; & q\delta(W\tilde{X}ED)_{j-1,\ell} + \alpha\beta d(W\tilde{X}(D+E))_{j-1,\ell} + \alpha\beta(W\tilde{X}D)_{j,\ell-1} \\
& -\gamma\delta^2 q^{|\tilde{X}|+1}(W\tilde{X}(D+E))_{j-2,\ell} - \gamma\delta q^{|\tilde{X}|}(W\tilde{X}D)_{j-1,\ell-1} \\
= \; & (\delta(W\tilde{X}DE)_{j-1,\ell} - \alpha\beta\delta(W\tilde{X}(D+E))_{j-1,\ell} + \gamma\delta^2 q^{|\tilde{X}|+1}(W\tilde{X}(D+E))_{j-2,\ell}) \\
& +\alpha\beta\delta(W\tilde{X}(D+E))_{j-1,\ell} + \alpha\beta(W\tilde{X}D)_{j,\ell-1} \\
& -\gamma\delta^2 q^{|\tilde{X}|+1}(W\tilde{X}(D+E))_{j-2,\ell} - \gamma\delta q^{|\tilde{X}|}(W\tilde{X}D)_{j-1,\ell-1} \\
= \; & \delta(W\tilde{X}DE)_{j-1,\ell} + \alpha\beta(W\tilde{X}D)_{j,\ell-1} - \gamma\delta q^{|\tilde{X}|}(W\tilde{X}D)_{j-1,\ell-1}.
\end{aligned}$$

In the first line above, note that we need only sum over i for $i \le j$. To deduce the second equation from the first, we use identity $2^{n-1}_{i,m'}$ for $i \le j$ and $m' \le m$. To deduce the third equation from the second, we use Lemma 6.5 to replace $D_{ijk\ell}$ by $qD_{i-1,j-1,k,\ell}$ in the first term. To deduce the fifth equation from the fourth, we apply identity $1^n_{j-1,m}$ to the term $q\delta(W\tilde{X}ED)_{j-1,\ell}$.

For the second implication, we have $X = \tilde{X}E$. Then $\beta(WXE)_{j-1,\ell}$ is equal to:

$$\begin{aligned}
&\delta(W\tilde{X}EE)_{j-1,\ell} = \delta\sum_{i,k}(W\tilde{X}E)_{ik}E_{i,j-1,k,\ell} \\
= \; & \sum_{i,k}\beta(W\tilde{X}D)_{i+1,k}E_{i,j-1,k,\ell} - \sum_{i,k}\alpha\beta(W\tilde{X})_{i+1,k-1}E_{i,j-1,k,\ell} + \gamma\delta q^{|\tilde{X}|}\sum_{i,k}(W\tilde{X})_{i,k-1}E_{i,j-1,k,\ell} \\
= \; & \beta q^{-1}(W\tilde{X}DE)_{j\ell} - \sum_{i,k}\alpha\beta(W\tilde{X})_{i+1,k-1}(q^{-1}\beta(D+E)_{i+1,j,k-1,\ell} + E_{i+1,j,k-1,\ell-1}) \\
& +\gamma\delta q^{|\tilde{X}|}\sum_{i,k}(W\tilde{X})_{i,k-1}(\beta(D+E)_{i,j-1,k-1,\ell} + qE_{i,j-1,k-1,\ell-1}) \\
= \; & \beta q^{-1}(W\tilde{X}DE)_{j\ell} - \alpha\beta^2 q^{-1}(W\tilde{X}(D+E))_{j\ell} - \alpha\beta(W\tilde{X}E)_{j,\ell-1} \\
& +\beta\gamma\delta q^{|\tilde{X}|}(W\tilde{X}(D+E))_{j-1,\ell} + \gamma\delta q^{|\tilde{X}|+1}(W\tilde{X}E)_{j-1,\ell-1} \\
= \; & (\beta(W\tilde{X}ED)_{j\ell} + \alpha\beta^2 q^{-1}(W\tilde{X}(D+E))_{j\ell} - \beta\gamma\delta q^{|\tilde{X}|}(W\tilde{X}(D+E))_{j-1,\ell}) \\
& -\alpha\beta^2 q^{-1}(W\tilde{X}(D+E))_{j\ell} - \alpha\beta(W\tilde{X}E)_{j,\ell-1} \\
& +\beta\gamma\delta q^{|\tilde{X}|}(W\tilde{X}(D+E))_{j-1,\ell} + \gamma\delta q^{|\tilde{X}|+1}(W\tilde{X}E)_{j-1,\ell-1} \\
= \; & \beta(W\tilde{X}ED)_{j\ell} - \alpha\beta(W\tilde{X}E)_{j,\ell-1} + \gamma\delta q^{|\tilde{X}|+1}(W\tilde{X}E)_{j-1,\ell-1}.
\end{aligned}$$

As before, in the sum in the first line above we need only to sum over i where $i \le j-1$. Then to go from the first equation to the second, we use identity $2^{n-1}_{i,m'}$, for $i \le j-1$ and $m' \le m$. In going from the second equation to the third, we replace $E_{i,j-1,k,\ell}$ by $q^{-1}E_{i+1,j,k,\ell}$ in the first term, using Lemma 6.5. We go from the fourth equation to the fifth by applying identity $1^n_{j,m-1}$ to the term $\beta q^{-1}(W\tilde{X}DE)_{j,\ell}$.

To prove the third implication, note that $(WXDE)_{j\ell} = \sum_{i,k}(WXD)_{ik}E_{ijk\ell}$ is equal to:

$$\begin{aligned}
& \beta^{-1}\delta\sum_{i,k}(WXE)_{i-1,k}E_{ijk\ell} + \alpha\sum_{i,k}(WX)_{i,k-1}E_{ijk\ell} - \beta^{-1}\gamma\delta q^{|X|}\sum_{i,k}(WX)_{i-1,k-1}E_{ijk\ell} \\
= \; & q\beta^{-1}d(WXEE)_{j-1,\ell} + \alpha\sum_{i,k}(WX)_{i,k-1}(\beta(D+E)_{i,j,k-1,\ell} + qE_{i,j,k-1,\ell-1}) \\
& -\beta^{-1}\gamma\delta q^{|X|}\sum_{i,k}(WX)_{i-1,k-1}(q\beta(D+E)_{i-1,j-1,k-1,\ell} + q^2E_{i-1,j-1,k-1,\ell-1}) \\
= \; & q\beta^{-1}\delta(WXEE)_{j-1,\ell} + \alpha\beta(WX(D+E))_{j\ell} + \alpha q(WXE)_{j,\ell-1} \\
& -\gamma\delta q^{|X|+1}\sum_{i,k}(WX(D+E))_{j-1,\ell} - \beta^{-1}\gamma\delta q^{|X|+2}(WXE)_{j-1,\ell-1} \\
= \; & (q(WXED)_{j\ell} - \alpha q(WXE)_{j,\ell-1} + q\beta^{-1}\gamma\delta q^{|X|+1}(WXE)_{j-1,\ell-1}) \\
& +\alpha\beta(WX(D+E))_{j\ell} + \alpha q(WXE)_{j,\ell-1} - \gamma\delta q^{|X|+1}\sum_{i,k}(WX(D+E))_{j-1,\ell} \\
& -\beta^{-1}\gamma\delta q^{|X|+2}(WXE)_{j-1,\ell-1} \\
= \; & q(WXED)_{j\ell} + \alpha\beta(WX(D+E))_{j\ell} - \gamma\delta q^{|X|+1}\sum_{i,k}(WX(D+E))_{j-1,\ell}.
\end{aligned}$$

In the first line above, note that we need only sum over $i \le j$. We then use $2^{n-1}_{i,m'}$ for $i \le j$ and $m' \le m$ to deduce the second line from the first. To go from the second to the third equation, we replace $E_{ijk\ell}$ in the first term by $qE_{i-1,j-1,k,\ell}$. To go from the fourth to the fifth equation, we use identity $2E^n_{j,m}$. □

6.4. **Applications.** Once we have a solution to the matrix ansatz, it is easy to express physical quantities in terms of matrix products [13]. Set $C = D + E$.

The partition function Z_n is written as $\langle W|C^n|V\rangle$, and the average particle number at site i, $\langle\tau_i\rangle_n$ (where the bracket indicates the average over the stationary probability distribution) is written as

$$\langle\tau_i\rangle = \frac{1}{Z_n}\langle W|C^{i-1}DC^{n-i}|V\rangle. \tag{4}$$

Similarly the two-point function $\langle\tau_i\tau_j\rangle_n$ is given by

$$\langle\tau_i\tau_j\rangle = \frac{1}{Z_n}\langle W|C^{i-1}DC^{j-i-1}DC^{n-j}|V\rangle, \tag{5}$$

and the n-point functions are expressed similarly. The particle current through the bond between the neighboring sites from left to right, which is defined by $J = \langle\tau_i(1-$

$\tau_{i+1}) - q(1-\tau_i)\tau_{i+1}\rangle$, is simply given by $J = \frac{Z_{L-1}}{Z_L}$. This expression is independent of i, as expected in the steady state.

Theorem 3.5 now follows from Theorem 6.2 and the expressions above for the current and m-point functions in terms of matrix products.

7. The proof of our Askey-Wilson moment formula

Before proving Theorem 4.1, we need to prove the following result.

Lemma 7.1. *Let D, E, W, V be a solution to the ansatz of Theorem 5.2, and let $\tilde{D}, \tilde{E}, \tilde{W}, \tilde{V}$ be a solution to the ansatz of Theorem 5.1. Let h denote the ratio $\frac{\tilde{W}\tilde{V}}{WV}$. Then if X is a word in D and E, and $\tilde{X}$ is the corresponding word in $\tilde{D}$ and $\tilde{E}$, then*

$$WXV = h^{-1}\tilde{W}\tilde{X}\tilde{V}\prod_{i=0}^{|X|-1}\lambda_i.$$

Proof. Let τ denote the type of X, and let $n = |X|$. We use induction on n. By Theorem 5.2 and Theorem 5.1 respectively, WXV and $\tilde{W}\tilde{X}\tilde{V}$ compute (unnormalized) steady state probabilities of being in state τ. Therefore $WXV = c_n\tilde{W}\tilde{X}\tilde{V}$ for some constant c_n that depends on n but not X. We want to show that $c_n = h^{-1}\prod_{i=0}^{n-1}\lambda_i$.

Since we have assumed that D, E, W, V satisfy the relations of Theorem 5.2, $\gamma WDXV - \alpha WEXV = \lambda_n WXV$. By induction, we conclude that

$$\gamma WDXV - \alpha WEXV = \lambda_n \tilde{W}\tilde{X}\tilde{V}h^{-1}\prod_{i=0}^{n-1}\lambda_i.$$

But also

$$\begin{aligned}\gamma WDXV - \alpha WEXV &= c_{n+1}\gamma\tilde{W}\tilde{D}\tilde{X}\tilde{V} - c_{n+1}\alpha\tilde{W}\tilde{E}\tilde{X}\tilde{V}\\ &= c_{n+1}(\gamma\tilde{W}\tilde{D}\tilde{X}\tilde{V} - \alpha\tilde{W}\tilde{E}\tilde{X}\tilde{V})\\ &= c_{n+1}\tilde{W}\tilde{X}\tilde{V},\end{aligned}$$

by Theorem 5.1. This shows that $c_{n+1} = h^{-1}\prod_{i=0}^{n}\lambda_i$, which completes the proof. □

We now prove Theorem 4.1, using some results of [39].

Proof of Theorem 4.1. Let $\tilde{Z}_L$ denote the partition function from [39], i.e. $\tilde{Z}_L = \tilde{W}(\tilde{D}+\tilde{E})^L\tilde{V}$, where $\tilde{D}, \tilde{E}, \tilde{W}, \tilde{V}$ are a solution to the ansatz of Theorem 5.1, and $\tilde{W}\tilde{V} = h_0$ (see [39, (4.18)]). Then by [39, Section 6.1],

$$\tilde{Z}_L = \oint_C \frac{dz}{4\pi iz} w\left((z+z^{-1})/2\right)\left[\frac{z+z^{-1}+2}{1-q}\right]^L.$$

Therefore

$$\tilde{Z}_L = \oint_C \frac{dz}{4\pi iz} w\left((z+z^{-1})/2\right)\left(\frac{2}{1-q}\right)^L\left[\frac{z+z^{-1}}{2}+1\right]^L,$$

which implies that

$$\left(\frac{1-q}{2}\right)^L \tilde{Z}_L = \oint_C \frac{dz}{4\pi i z} w\left((z+z^{-1})/2\right)\left[\frac{z+z^{-1}}{2}+1\right]^L$$
$$= \sum_{k=0}^{L}\binom{L}{k}\oint_C \frac{dz}{4\pi i z} w\left((z+z^{-1})/2\right)\left[\frac{z+z^{-1}}{2}\right]^k$$
$$= \sum_{k=0}^{L}\binom{L}{k}\mu_k.$$

Inverting this, we get

$$\mu_k = \sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\left(\frac{1-q}{2}\right)^{\ell}\tilde{Z}_\ell.$$

By Theorem 3.4, we know that Z_ℓ is the generating function for all staircase tableaux of size ℓ. By Lemma 7.1, $Z_\ell = h_0^{-1}\tilde{Z}_\ell \prod_{i=0}^{\ell-1}(\alpha\beta - \gamma\delta q^i)$. Therefore

$$\mu_k = h_0 \sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\left(\frac{1-q}{2}\right)^{\ell}\frac{Z_\ell}{\prod_{i=0}^{\ell-1}(\alpha\beta-\gamma\delta q^i)}.$$

□

8. Open problems

8.1. **Symmetries in the ASEP.**
Recall that the ASEP has "left-right," "arrow-reversal," and "particle-hole" symmetries, which imply Observation 2.2.

Problem 8.1. *For each symmetry above, prove the corresponding identity in Observation 2.2 by describing an appropriate involution on staircase tableaux.*

Proving the second identity in this manner is easy. Namely, define a map ι by letting $\iota(\mathcal{T})$ be the tableau obtained from $\mathcal{T}$ by switching β's and δ's, and switching α'a and γ's; clearly if $\mathrm{wt}(\mathcal{T}) = \alpha^{i_1}\beta^{i_2}\gamma^{i_3}\delta^{i_4}q^{i_5}u^{i_6}$, then $\mathrm{wt}(\iota(\mathcal{T})) = \alpha^{i_3}\beta^{i_4}\gamma^{i_1}\delta^{i_2}q^{i_6}u^{i_5}$. This plus Theorem 3.4 proves the second identity. It remains to find an involution ι' proving the first identity (the remaining involution can be constructed by composing ι' with ι). A natural guess is to define $\iota'(\mathcal{T})$ by transposing $\mathcal{T}$ then switching α's and δ's, and β's and γ's. This works when $q = t$, but not for $q \neq t$.

8.2. **Lifting the ASEP to a Markov chain on staircase tableaux.**

Question 8.2. *Find a Markov chain on the set of all staircase tableaux of size n which* projects *to the ASEP in the sense of* [9]*, such that the steady state probability of a tableau $\mathcal{T}$ is proportional to* $\mathrm{wt}(T)$*. Such an approach would give a completely combinatorial proof of Theorem 3.4. (This was done in* [9] *for $\gamma = \delta = 0$.)*

8.3. **Keeping track of where the particles come from.** In the ASEP, a black particle enters from either the left (at rate α) or from the right (at rate δ). Similarly, a "hole" (or a white particle) enters from either the left (at rate γ) or from the right (at rate β). We might therefore hope to define a more refined Markov chain on 4^n states, which projects to the ASEP and keeps track of where the black and white particles came from. Indeed, the staircase tableaux themselves seem to be keeping track of more information than just the colors of the particles.

Problem 8.3. *Fix a lattice of n sites, and define a Markov chain on 4^n states (words of length n on the alphabet $\{\alpha, \beta, \gamma, \delta\}$) with the following properties:*

- *particles labeled α and γ always enter the lattice from the left, and particles labeled β and δ always enter from the right;*
- *the Markov chain projects to the ASEP;*
- *the steady state probability of state $(\tau_1, \dots, \tau_n)$ is proportional to the generating function for all staircase tableaux whose border is $(\tau_1, \dots, \tau_n)$.*

8.4. **A combinatorial proof of the relations of the ansatz.** In Section 6, we gave a combinatorial proof that D, E, V, W satisfy relation (III) of Theorem 5.2, by translating it into a statement about tableaux. However, we have not yet found a combinatorial proof that D, E, V, W satisfy (I) and (II).

Problem 8.4. *Give a combinatorial proof of relations (I) and (II) of Theorem 5.2.*

We note that when $q = t$, or one of $\alpha, \beta, \gamma, \delta$ is 0, the above problem is easy.

8.5. **Specializing our moment formula for Askey-Wilson polynomials.**

Problem 8.5. *Show directly that our moment formula recovers already-known moment formulas for specializations or limiting cases of Askey-Wilson polynomials.*

9. Appendix: Staircase, permutation, and alternative tableaux

Definition 9.1. [37, 29] *A* permutation tableau $\mathcal{T}$ *is a Young diagram (where rows may have length 0) whose boxes are filled with 0's and 1's, such that each column contains at least one 1, and there is no 0 which has simultaneaously a 1 above it in the same column and a 1 to its left in the same row. The* length *of $\mathcal{T}$ is the sum of its number of rows and columns.*

Definition 9.2. [41] *An* alternative tableau $\mathcal{T}$ *is a Young diagram (where rows* and *columns may have length 0) whose boxes are either empty or filled with left arrows $\leftarrow$ or up arrows $\uparrow$, such that all boxes to the left of a $\leftarrow$ are empty and all boxes above an $\uparrow$ are empty. The* length *of $\mathcal{T}$ is the sum of its number of rows and columns.*

See [28] for more information about alternative tableaux.

Proposition 9.3. *There is a bijection between staircase tableaux of size n which do not contain any γ or δ, and:*

(1) *permutation tableaux of length* $n+1$*;*
(2) *alternative tableaux of length* n*.*

Proof. We first give a bijection from permutation tableaux to staircase tableaux. Define a *restricted* 0 of a permutation tableau to be a 0 which has a 1 above it in the same column. A restricted 0 is *rightmost* if it is the rightmost restricted 0 in its row. If $\mathcal{T}$ is a permutation tableau, we replace with a $\leftarrow$ every rightmost restricted 0, and replace with a $\uparrow$ every 1 which is the highest 1 in its column but is not in the top row. We replace every other entry of $\mathcal{T}$ by an empty box, and delete the top row (but we remember the length of the top row by possibly inserting empty columns to the right). The result is an alternative tableau, see Figure 5, and the map can be easily inverted.

For the second bijection, fix a staircase tableau of size n. For i from 1 to n, if the ith diagonal box contains an α, then delete this entry and the column above it (this α will correspond to a vertical step in the south-east border of the resulting alternative tableau). Otherwise if the ith diagonal box contains a β, delete this entry and the row to its left (this β will correspond to a horizontal step in the south-east border of the resulting tableau). Then replace each α with an $\uparrow$ and each β with a $\leftarrow$, and discard all the other entries. See Figure 5. □

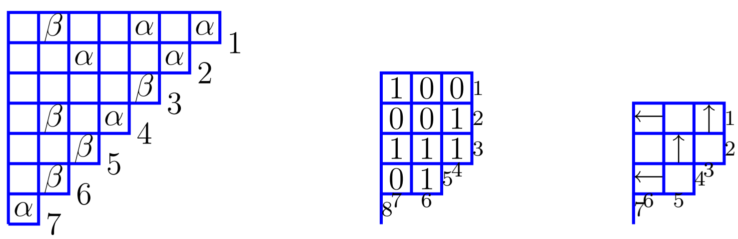

Figure 5. From a staircase tableau, to a permutation tableau and an alternative tableau

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