MATH 277 PROBLEM SET 1
(DUE TUESDAY, SEPTEMBER 30)

Do (at least) six out of the following eight problems. If you do more than six problems, circle the problems which you want graded.

(1) Let V be a finite-dimensional irreducible module for $\mathfrak{s}l_2$ with highest weight λ, and let V_μ denote the μ-weight space. Show that $e \cdot V_\mu \subset V_{\mu+2}$ and $f \cdot V_\mu \subset V_{\mu-2}$. Let v_0 denote a highest weight vector for V (a vector such that $e \cdot v_0 = 0$ and $h \cdot v_0 = \lambda v_0$) and define $v_j = \frac{1}{j!} f^j \cdot v_0$ for $j \geq 0$. Show that $h \cdot v_j = (\lambda - 2j) v_j$, $f \cdot v_j = (j+1) v_{j+1}$. Explain why this implies that the v_j's provide a basis for V.

(2) Prove that the Lie algebra $\mathfrak{g} = \mathfrak{s}\ell_n$ of traceless $n \times n$ matrices with the usual bracket $[A, B] = AB - BA$ is a Kac-Moody algebra. Of what type is it?

(3) If U and W are representations of a Lie algebra $\mathfrak{g}$, we define the action of $\mathfrak{g}$ on $U \otimes W$ by:

$$X \cdot (v \otimes w) = X \cdot v \otimes w + v \otimes X \cdot w.$$

(This is in order to be compatible with the standard action of the corresponding Lie group on the tensor product.) Check that this indeed defines an action on the tensor product. Now let V be the standard two-dimensional representation of $\mathfrak{s}\ell_2$ with basis e_1 and e_2. Describe the eigenvalues of h on the most natural basis for $\mathrm{Sym}^n(V)$. What does this tell you about $\mathrm{Sym}^n(V)$? Finally, compute the character of $\mathrm{Sym}^n(V)$ using the Weyl character formula. (In the case of a semisimple Lie algebra ρ is half the sum of the positive roots.)

(4) Let V be as in Problem 3 above. By considering the eigenvalues of h on $\mathrm{Sym}^2(V) \otimes \mathrm{Sym}^5(V)$ (and not using crystals), describe how this representation decomposes into irreducibles. Then perform the same computation, this time using crystals. Give the general formula for the decomposition of $V(m) \otimes V(n)$ where $V(m)$ is the irreducible representation of dimension $m+1$.

(5) Let $L = \mathfrak{s}\ell_2$ and $M = \mathfrak{s}\ell_3$. M contains a copy of L in its upper left-hand 2×2 position. Show that M is a direct sum of irreducible L-submodules (when we view M as an L-module via the adjoint representation), and that it decomposes as $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.

(6) Recall that a *root system* is defined as follows. We fix a Euclidean space E. Then R is a root system if and only if the following properties hold:
   (a) R is a finite set spanning E.
   (b) $\alpha \in R$ implies $-\alpha \in R$, but $k\alpha$ is not in R if k is any real number other than ± 1.
   (c) For $\alpha \in R$, the reflection W_α in the hyperplane $\alpha^\perp$ maps R to itself.
   (d) For $\alpha, \beta \in R$, the real number $n_{\beta\alpha} = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer.

   Classify the root systems of rank 2.

(7) Give a brief (one-half to one page) description of the connection between root systems and Lie algebras, including the definition of the Killing form.

(8) For a semisimple complex Lie algebra $\mathfrak{g}$ and representation V, a nonzero vector $v \in V$ that is both an eigenvector for the action of the Cartan subalgebra $\mathfrak{h}$ and in the kernel of $\mathfrak{g}_\alpha$ for all $\alpha \in R^+$ is called a *highest weight vector* of V. For such $\mathfrak{g}$, prove that:
   (a) every finite-dimensional representation V of $\mathfrak{g}$ possesses a highest weight vector.
   (b) the subspace W of V generated by the images of a highest weight vector v under successive applications of root spaces $\mathfrak{g}_\beta$ for $\beta \in R^-$ is an irreducible subrepresentation.
   (c) an irreducible representation possesses a unique highest weight vector up to scalars.