MATH231BR: ADVANCED ALGEBRAIC TOPOLOGY – PAPER TOPICS

1. Details about the papers

The goal of the midterm and final papers is to produce nice expository articles about topics in algebraic topology or related fields in a broad sense. You learn something from it, while the broader mathematical community benefits from the time you have spent understanding and summarizing some difficult topic in a field where the literature is often lacking. Hence the main goal is produce understandable and engaging writing, which draws attention to the important ideas and subtleties. You can refer to references for the details, and hence should not feel obliged to give full proofs (though of course it would be nice if you could include them).

The midterm paper should be 4–8 pages, and the term paper 8–16 pages. They should be in LaTeX, preferably be handed in electronically, and you should not collaborate with other students. You will have to hand in a topic and rough draft before handing in the final papers, so I can give comments about its suitability and suggestions for references.

1.1. Important dates.

- Midterm paper topic due: Thursday, February 22nd.
- Midterm paper rough draft due: Thursday, March 1st.
- Midterm paper due: Friday, March 9th (2:00pm).
- Final paper topic due: Thursday, April 19th.
- Final paper rough draft due: Thursday, April 26th.
- Final paper due: Friday, May 11th (2:00pm).

2. Topic suggestions

Below I give some suggestions for topics, and the ones that I think are more suitable for a final paper have been marked with an asterisk. These topics may be too broad, so you might want to select among the different aspects of a topic. Feel free to pick a different topic, and I will comment on your suggestion when you submit it.

2.1. Point-set topology.

- * Discuss one of the other cohomology theories, e.g. Cech cohomology, Alexander-Spanier cohomology, or sheaf cohomology. When do they coincide with singular cohomology? See e.g. [Bre97].
- Discuss Barratt-Milnor's paper on anomalous singular homology [BM62].
- Discuss some of the point-set topology of (closed) cofibrations, e.g. Lillig's union theorem [Lil73] or Strøm-Kieboom's pullback theorem [Kie87].
- Discuss topics from the theory of decomposition spaces and Bing topology, see e.g. [Fre13]. This eventually led to the proof of the 4-dimensional topological h-cobordism theorem [FQ90].

2.2. Classical homotopy theory.

- Discuss Phantom maps, see e.g. Sections 9.5 and 25.12.2 of [Str11].
- Discuss the plus construction and its universal property, see e.g. [Ger73].
- * Discuss the Bockstein spectral sequence and Browder's generalization of the fact that $\pi_2(G) = 0$ for G a simply-connected Lie groups. See Chapter 10 of [McC01].

2.3. Homotopy theory and homotopical algebra.

- What is a quasi-category? See Chapter 1 of [Lur09].
- * Explain the relationship between homotopy (co)limits and quasi-categorical (co)limits, see Chapter 4 of [Lur09].
- * Give an exposition why we can think of a complete Segal space as a model for ∞-categories. Explain the relationship with quasi-categories.
- Explain Lewis' argument that there is no convenient category of spectra [Lew91]. How does this relate to different model structures on the category of symmetric/orthogonal spectra (in particular absolute and positive model structures)?
- Describe the theory of left Bousfield localization of model categories [Hir03].
- Describe the Dwyer-Kan hammock localization and its relation to simplicial model categories, see e.g. [DK80] among many.
- * Explain how cubical sets with connection are related to simplicial sets [Mal09]. How does this fit into the work of Cisinski [Cis06]?
- * Discuss the category of quasi-topological spaces. Is there a model structure such that it is Quillen equivalent to Top with either the Serre or Hurewicz model structure?

2.4. Localized homotopy theory.

- Discuss Sullivan's CDGA approach to rational homotopy theory [Sul77].
- Discuss Quillen's DGLA approach to rational homotopy theory [Qui69].
- * Discuss the co-algebraic approach to rational homotopy theory.
- Explain how to compute the rational homotopy type of mapping spaces or section spaces, see e.g. [Hae82].
- Discuss the Hilton-Roitberg examples as an application of Zabrodsky mixing. See Section 2.11 of [Nei10].
- * Discuss the Sullivan conjecture and its proof [Sch94].
- * What is the relationship between $H\mathbb{Z}$-localization and the plus construction?

2.5. Geometric applications.

- Discuss the Spivak normal fibration of a Poincaré duality space. Explain how this fits into surgery theory for high-dimensional manifolds [Wal99].
- State and prove the Dold-Thom theorem. See Section 25.10 of [Str11] how to deduce the Künneth theorem from this.
- Discuss the Lusterik-Schnirelmann category of a space and its basic properties, see e.g. Section 9.7 of [Str11] or [CLOT03]. This is often studied using rational homotopy theory.
- Give the precise statement of the Atiyah-Singer index theorem [LM89].
- Smith theory for finite group actions on nice spaces.

2.6. Homotopy groups of spheres.

- * Discuss the EHP spectral sequence and do a few computations with it, see e.g. Chapter 1.5 of [Rav86].
- * Discuss Jamess 2-primary exponent theorem $4^n \pi_k(S^{2n+1})_{(2)} = 0$ for $k > 2n+1$. Toda generalized it to odd primes. See [Nei10].
- * Explain the May spectral sequence and how it may be used to compute the E^2-page of the Adams spectral sequence. How far can you get with it? See Chapter 3.2 of [Rav86].
- * What is a Toda bracket [Tod62]?

2.7. Stable homotopy theory.

- Describe elliptic cohomology and its application to Ochanine's theorem, see Chapter 8 of [HBJ92].
- What is TMF? See e.g. [DFHH14].

- * Give a conceptual outline of stable chromatic homotopy theory [Rav86].
- * Mahowald's theorem on expressing the Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$ as a Thom spectrum. There is a variety of generalizations, see e.g. [CMT81].
- * Discuss equivariant stable homotopy theory.
- * Discuss motivic stable homotopy theory.

2.8. Characteristic classes.

- What is the Todd genus [Hir95]?
- What is the $\hat{A}$-genus? Explain its application to positive scalar curvature metrics.
- What is the L-genus?
- What is the Witten genus?
- * Explain the number-theoretic result of [BB17] on the coefficients of the L-polynomials. Does a similar result hold for the Todd genus, elliptic genus or Witten genus?
- Explain the Chern-Weil construction of characteristic classes, see e.g. Chapter 19 of [Hus94] or Chapter 5 of [CC03]. See Chapter 6 of [CC03] for application to foliations.
- Discuss the (generalized) MMM-classes [Mor01], which are characteristic classes for manifold bundles. What do the Madsen-Weiss theorem [MW07] and its generalizations say about these [GTMW09]?
- Discuss Stiefel-Whitney classes for real representations of finite groups [GKT89]. Do some computations. How are they related to Stiefel-Whitney classes of vector bundles? What about Chern classes for complex representations?

2.9. K-theory.

- * The relationship between K-theory and Clifford algebras (e.g. Chapter 12 of [Hus94]), possibly a proof of Bott periodicity using Clifford-equivariant Fredholm operators [AS69].
- Adams operations on K-theory and λ-rings (e.g. Chapter 13 of [Hus94]). There is an extension to algebraic K-theory, e.g. Chapter IV.5 of [Wei13].
- * The K-theoretic proof of the non-existence of elements of Hopf invariant one (note that this uses Adams operations), see e.g. Chapter 15 of [Hus94] or [AA66].
- * A classical application of K-theory is to the problem of the existence of vector fields on spheres, solved by Adams. This involves the study of the image of J, see e.g. Chapter 16 of [Hus94].
- Discuss how the e-invariant gives lower bounds on the image of J, see Section 4.1 of [Hat03].
- Give Atiyah's elementary proof of Bott periodicity, see e.g. Section 2.2 of [Ati67] and Sections 2.1 and 2.2 of [Hat03].
- Describe "real" K-theory KR [Ati66]. What are its applications?
- * Construct the Atiyah-Bott-Shapiro orientation $MSpin \to KO$ as a map of spectra.

2.10. Cobordism.

- Describe bordism with Baas-Sullivan singularities and its use in defining spectra relevant to chromatic homotopy theory [Baa73].
- State the computation of the homotopy type of the cobordism category [GTMW09]. Do some computations with $MTO(d)$ or $MTSO(d)$ spectra.
- Explain the proof of Genauer's theorem on the homotopy type of the cobordism category with boundary [Gen12].
- * Discuss the Stolz-Teichner program on constructing ordinary cohomology and topological K-theory through Euclidean field theories [ST11].

References

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