ERRATUM TO: THREE APPLICATIONS OF DELOPPING TO $h$-PRINCIPLES

ALEXANDER KUPERS

Abstract. We correct a mistake in Three applications of delooping to $h$-principles, though it does not affect any of the results.

1. Lemma 20 is incorrect

Johannes Ebert has pointed out that Lemma 20 of [Kup18] is incorrect. This lemma said:

Lemma 1.1. Let $b \in \mathcal{B}_\Psi(S^{n-1})$, and suppose that $\Psi$ satisfies condition (H). Then $\Psi(D^n_+ \text{rel } b) \neq \emptyset$ if and only if $\Psi(D^n_- \text{rel } b) \neq \emptyset$.

A counterexample is provided by letting $\Psi$ be the sheaf which assigns to $U$ the set of everywhere non-zero sections of the tangent bundle $TU$; that is, the space of everywhere non-zero vector fields on $U$. This provides a counterexample because it is not always possible to extend an everywhere non-zero section of $TS^n$ over a hemisphere $D^n_+$ to the entire sphere $S^n$. The easiest obstruction is the non-vanishing of the Euler characteristic.

The mistake in the proof occurs at the very start: the statement of Lemma 22 is equivalent to $\Psi(D^n_+ \text{rel } b) \neq \emptyset$ if and only if $\Psi(D^n_- \text{rel } b') \neq \emptyset$ where $b'$ is the boundary condition obtained by applying the diffeomorphism $D^n_+ \cong D^n_-$. After having passing to $\Psi^f$, this amounts to modifying the map $S^{n-1} \rightarrow \Psi^f(\mathbb{R}^n)$ by applying the map induced on the frame bundle by the bundle isomorphism

$$D^n_+ \times \mathbb{R}^n \rightarrow TS^n|_{D^n_+} \rightarrow TS^n|_{D^n_-} \rightarrow D^n_- \times \mathbb{R}^n,$$

a composition of reflection and the clutching map of $TS^n$. Thus the proof goes through only if the clutching map can be taken to the identity, i.e. $n = 1, 3, 7$.

2. Modifications to the paper

Lemma 20 does not play a major role in the paper; its purpose was to simplify arguments by removing the distinction between boundary conditions $b \in \mathcal{B}_\Psi(S^{n-1})$ which extend over $D^n_+$ (“right-fillable”) and those which extend over $D^n_-$ (“left-fillable”). Indeed, the results of the paper hold without substantial modification.

Firstly, the reader interested in the applications may replace Lemma 20 by the assumption that for $\Psi$ a boundary condition is left-fillable if and only if it is right-fillable. This is satisfied in all examples in which a homotopical $h$-principle is proved:

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Lemma 57 concerns spaces of functions with moderate singularities. Here a boundary condition \( b \in \mathcal{B}_f(-, D)(S^{n-1}) \) is left-fillable if and only if it is underlying continuous map \( f_b : S^{n-1} \to Z \) is null-homotopic if and only if it is right-fillable.

Lemma 73 concerns framed functions, and Igusa’s argument gives that every boundary condition is both left- and right-fillable.

Secondly, the proofs can easily be modified to remove any reliance on Lemma 20. Here the required changes:

**Definition 19:** This needs to be replaced with the following definition of a fillable boundary condition: \( b \in \mathcal{B}_\Psi(S^{n-1}) \) is **fillable** if it extends to \( D_n^+ \).

**Lemma 20:** This needs to be replaced with the statement that if \( \Psi \) satisfies condition (H), then if \( b \) is fillable and there is a morphism \( b \to b', b' \) is also fillable. Thus the fillable boundary components form a collection of path-components of \([\mathcal{C}_\Psi]\).

This is proven by first using the homological \( h \)-principle to pass to \( \Psi^f \) and then observing that if there is a homotopy from a section \( s_0 \in (TS^{n-1} \oplus \epsilon) \times_{GL_n(\mathbb{R})} \Psi(\mathbb{R}^n) \) to another section \( s_1 \), there is also a homotopy of sections from \( s_1 \) to \( s_0 \).

**Proof of Theorem 27:** In this proof we use an embedding \( \mathbb{R}^n \hookrightarrow M \setminus A \), from which we derive an embedding \( S^{n-1} \times \mathbb{R} \hookrightarrow M \setminus A \): by convention, we do so by composing with an embedding \( S^{n-1} \times \mathbb{R} \hookrightarrow \mathbb{R}^n \) with the property that each \( S^{n-1} \times \{t\} \) bounds a disk in \( \mathbb{R}^n \) to the right. This guarantees that all boundary conditions which appear are fillable in the above sense.

In any other place where “fillable boundary conditions” are mentioned, no modification is needed.

**Remark 2.1.** There is no mathematical reason to define a fillable boundary condition to be one which extends to \( D_n^+ \) rather than one which extends to \( D_n^- \). This is a matter of convention.

**References**