

The structure of Donaldson's invariants for four-manifolds not of simple type

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1. Introduction

In [2], a theorem was proved which described the structure of Donaldson's polynomial invariants for 4-manifolds satisfying the 'simple type' condition. At present, there is no example of a 4-manifold with $b_2^+ \geq 3$ which does not have simple type. On the other hand, there is no argument which suggests that such examples do not exist.

It is reasonable to ask what the counterpart of the theorem of [2] might be, for a 4-manifold which did not have simple type. In April 1994, the authors spent a month at Princeton, trying to find an answer. The tools at hand were the techniques developed in [2] and the general blow-up formula, in the form given by Fintushel and Stern in [1].

An answer of sorts was obtained, but there were some shortcomings in the mathematical argument. In [2] it was shown that, whenever a 4-manifold contained an embedded surface Σ of reasonably large genus g and self-intersection number n , then there were universal relations among the Donaldson invariants, in a range determined g and n (the bigger n and g are, the better). We obtained a structure theorem for manifolds not of simple type only by using these relations beyond the range in which they had been proved to be valid (for small g and n). For this reason alone, the results remain conjectural.

At that time, we made a note of the outcome. This was distributed to anybody who asked for it. More recently, interest in the question has been renewed, largely because of Witten's conjecture [3] relating the Donaldson invariants to the Seiberg-Witten invariants. This paper is a tidied-up version of the note we produced in April 1994. We wish to emphasize that *all* the results presented are conjectural (for

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the reason just described); and even if they are correct, it seems unlikely that we have stated them in their most natural form.

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2. The formula: a first pass

Let X be a 4-manifold with $b_1 = 0$ and $b_2^+ \geq 3$ whose polynomial invariants (for our convenience) have degree $0 \pmod{4}$ as polynomials on the second homology. Let x be the point class, let Σ be a surface representing an arbitrary 2-dimensional homology class. Let Σ have self-intersection $n > 0$, and put $a = 2g - 2 - n$ (g being the genus). We suppose n (and hence a) is even and g is odd. Our notation follows [2].

The first thing is the 4-dimensional class coming from the point-class x . We conjecture (see the Introduction) that x satisfies the relation $(x^2 - 4)^\tau = 0$ for $\tau = [a/4] + 1$ (with $a = a(\Sigma)$ as above). That is,

$$D((x^2 - 4)^\tau z) = 0$$

for all z . We say X has type (less than or equal to) τ . The case $\tau = 1$ is the simple type condition.

Next, we conjecture that there is the following structure theorem for the invariants, restricted to the classes of x and Σ . We encode the Donaldson invariant as a function of two variables t and λ , and we have

$$\begin{aligned} D(e^{t\Sigma + \lambda x}) &= e^{2\lambda} \Lambda(\lambda) L(2t; \mathbf{X})^{-n/2} \sum_{r=0}^{a/2} \text{nd}(2t; \mathbf{X})^r \text{cd}(2t; \mathbf{X})^{a/2-r} \beta_r \\ &+ e^{-2\lambda} \Lambda(-\lambda) L(2t; -\mathbf{X})^{-n/2} \sum_{r=0}^{a/2} \text{cd}(2t; -\mathbf{X})^r \text{nd}(2t; -\mathbf{X})^{a/2-r} \beta_r. \end{aligned} \tag{1}$$

This formula needs some explanation. The Jacobian elliptic functions and the Theta functions are functions of two variables: the second is most often taken to be the ‘modulus’ k . We regard them as functions $\text{cd}(t; X)$ etc, where the (complex) parameter X is related to k by $X = 4k^2 - 2$. In (1), we have put a matrix \mathbf{X} into the power series in the place occupied by X . We set

$$\mathbf{X} = 2\mathbf{I} + \mathbf{N},$$

a standard (upper) Jordan block of rank $\tau \geq 1$, we set

$$\Lambda(\lambda) = \left(1, \lambda, \frac{\lambda^2}{2}, \dots, \frac{\lambda^{\tau-1}}{(\tau-1)!}\right),$$

and the β_r are arbitrary column vectors of length τ (depending on Σ – these are the arbitrary constants). There is a restriction on the β_r however: the bottom $[(r+1)/2]$ entries of β_r are zero. The function L is given by

$$L(t; X) = e^{(1-2\Theta''(0)/\Theta(0))t^2/4} \begin{pmatrix} \Theta_1(t) \\ \Theta_1(0) \end{pmatrix},$$

The function Θ_1 is the theta function

$$\Theta_1(u) := \theta_3(v) = \sum_{m=-\infty}^{\infty} q^{m^2} e^{2imv},$$

where $v = \pi u/2K$ and $q = e^{-\pi K'/K}$. This q is a function of k^2 (and hence of X) through $K = K(k)$, which is the complete elliptic of the first kind:

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$K' = K(k')$$

$$k^2 + (k')^2 = 1.$$

The theta function Θ is

$$\Theta(u) := \theta_4(v) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m^2} e^{2imv}.$$

When X has simple type, only the top entry of each β_r is non-zero: we can replace τ by 1. So \mathbf{X} becomes 2, Λ becomes 1 and so on. The function $\text{nd}(2t; 2)$ is $\cosh(2t)$ and $\text{cd}(2t, 2) = 1$. Similarly, $\text{nd}(2t, -2) = 1$ and $\text{cd}(2t, -2) = \cos(2t)$. The expression reduces to

$$D(e^{t\Sigma+\lambda x}) = e^{2\lambda} e^{nt^2/2} \sum_{r=0}^{a/2} \beta_r \cosh(2t)^r$$

$$+ e^{-2\lambda} e^{-nt^2/2} \sum_{r=0}^{a/2} \beta_r \cos(2t)^r. \quad (2)$$

There is an important point concerning the way this changes when you scale Σ . Let Σ_2 be an embedded surface representing the homology class 2Σ and having

genus determined by the relation $a(\Sigma_2) = 2a(\Sigma)$. Applying the structure theorem (1) to Σ_2 one obtains a different expression for the same function of λ and t : whereas (1) involves a homogeneous polynomial of degree $a(\Sigma)/2$ in $\text{nd}(2t)$ and $\text{cd}(2t)$, the new expression is a polynomial of degree $a(\Sigma)$ in $\text{nd}(t)$ and $\text{cd}(t)$. In the simple type case, the double-angle formulae provide the explanation. But for the elliptic functions, it is not possible to express, say, $\text{nd}(2t; X)^p \text{cd}(2t; X)^q$ as a polynomial of degree $2(p+q)$ in $\text{nd}(t; X)$ and $\text{cd}(t; X)$. The way out of this quandry is the observation that it is possible to write $\text{nd}(2t; X)^p \text{cd}(2t; X)^q$ this way *up to terms of order* $(X-2)^{\lfloor p/2 \rfloor}$ *inclusive* when one expands these functions of two variables as a Taylor series in $(X-2)$ around the point $X=2$ (the point at which they become elementary functions). This observation is the basis of the proposed proof that (x^2-4) is ‘nilpotent’.

3. A closer look

The formula (1) is slightly cryptic. What is really involved is the power-series expansion of $\text{cd}(2t; X)$ etc. around $X=2$, as far as the term in $(X-2)^{\tau-1}$. These expansions involve only elementary functions: polynomials in $\cos(2t)$, $\cosh(2t)$, $\sin(2t)$, $\sinh(2t)$ and t . For example, we have

$$\text{cd}(2t; X) = 1 - \frac{1}{8}(X-2)(1 - \cosh^2(2t)) + O(X-2)^2.$$

So, when $\tau = 2$,

$$\text{cd}(2t; \mathbf{X}) = \mathbf{I} - \frac{1}{8}(1 - \cosh^2(2t))\mathbf{N}$$

and

$$\Lambda(\lambda) \text{cd}(2t; \mathbf{X})\beta = a + b\lambda + (b/8)(1 - \cosh^2(2t))$$

when $\beta = (a, b)^T$. Thus the form of (1) in the case $n=0$ and $a=4$ (which is not really an applicable example, since n should really be positive) is

$$\begin{aligned} & e^{2\lambda} (a_0 + b_0\lambda + (b_0/4) \sinh^2(2t) + a_1 \cosh(2t) + a_2 \cosh^2(2t)) \\ & + e^{-2\lambda} (a_0 + b_0\lambda - (b_0/4) \sin^2(2t) + a_1 \cos(2t) + a_2 \cos^2(2t)) \end{aligned}$$

which can be recast as

$$\begin{aligned} D(e^{t\Sigma + \lambda x}) &= e^{2\lambda} (c_0 + b_0\lambda + c_1 \cosh(2t) + c_2 \cosh(4t)) \\ &+ e^{-2\lambda} (c_0 - b_0\lambda + c_1 \cos(2t) + c_2 \cos(4t)). \end{aligned}$$

The structure in this example is therefore exactly as in the simple type case, except that there is an extra term $b_0\lambda$. For larger values of τ , the picture is not quite

so simple, but nearly so. When one neglects terms of order $(X - 2)^\tau$, it seems that the formula (1) can be recast as

$$D(e^{t\Sigma+\lambda x}) = e^{2\lambda} \Lambda(\lambda) e^{nt^2 M(\mathbf{X})/2} \sum_{r=0}^{a/2} \text{nd}(2rt; \mathbf{X}) \beta'_r + e^{-2\lambda} \Lambda(\lambda) e^{nt^2 M(-\mathbf{X})/2} \sum_{r=0}^{a/2} \text{cd}(2rt; -\mathbf{X}) \beta'_r, \quad (3)$$

with

$$M(X) = 1 - 2E(X)/K(X) = 2\Theta''(0)/\Theta(0) - 1,$$

and $E(X)$ and $K(X)$ are the complete elliptic integrals of the second and first kind, regarded as functions of X again. In writing the formula this way, we have used the ‘approximate’ multi-angle formulae. The vectors β'_r are linear combinations of the β_r and satisfy the same constraints. The function $M(X)$ takes the values 1 and -1 at $X = 2$ and $X = -2$.

There is another way to express these formulae, without matrices. If β is a vector, as above, and $b(\lambda)$ is the polynomial obtained as the scalar product $\Lambda(\lambda)\beta$, then we have

$$\Lambda(\lambda)\mathbf{X}\beta = 2b(\lambda) + \frac{d}{d\lambda}b(\lambda),$$

$$\Lambda(-\lambda)(-\mathbf{X})\beta = -2b(-\lambda) + \frac{d}{d\lambda}b(-\lambda).$$

Exploiting this way of writing the equations has the advantage that τ need not be mentioned a priori. Putting this together, it seems that the structure theorem for the invariants of our 4-manifold should read:

Conjecture. *There exist finitely many two-dimensional cohomology classes K_1, \dots, K_s and polynomials $P_1(\lambda), \dots, P_s(\lambda)$ such that, for all $h \in H_2(X)$ we have*

$$D(e^{h+\lambda x}) = \exp\left(\frac{Q(h)}{2}M\left(\frac{d}{d\lambda}\right)\right) \left(\sum_{i=0}^s \text{nd}(K_i(h); \frac{d}{d\lambda})(e^{2\lambda} P_i(\lambda)) + \sum_{i=0}^s \text{cd}(K_i(h); \frac{d}{d\lambda})(e^{-2\lambda} P_i(-\lambda)) \right).$$

Furthermore, if Σ_g is any embedded surface of positive square, we have the inequality

$$2g - 2 \geq Q(\Sigma) + K_i(\Sigma) + 4 \deg P_i$$

for all i .

This conjecture would be proved were it not for the difficulties mentioned in the introduction. At the time of writing, it would be natural to conjecture that the coefficients of the P_i can be derived from the Seiberg–Witten invariants. Such a conjecture could not have been made (at least by these authors) in April 1994.

References

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