

RECURRENCE RELATIONS AND ASYMPTOTICS FOR FOUR-MANIFOLD INVARIANTS

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ABSTRACT. The polynomial invariants q_d for a large a class of smooth 4-manifolds are shown to satisfy universal relations. The relations reflect the possible genera of embedded surfaces in the 4-manifold, and lead to a structure theorem for the polynomials. As an application, one can read off a lower bound for the genera of embedded surfaces from the asymptotics of q_d for large d . The relations are proved using moduli spaces of singular instantons.

1. INTRODUCTION

Donaldson's polynomial invariants [3] are invariants of smooth, oriented, compact 4-manifolds X without boundary. They and their close cousins are the only useful invariants we have at present to distinguish different 4-manifolds of the same homotopy type. We recall that when X is simply connected and $b^+(X)$ is odd and not less than three, the polynomial invariants take the form of homogeneous polynomial functions

$$q_d : H_2(X, \mathbb{R}) \rightarrow \mathbb{R}.$$

Here b^+ is the dimension of a maximal positive subspace for the intersection form on $H_2(X)$. The index d is the degree of q_d and the invariants are defined for all non-negative degrees $d \equiv \frac{1}{2}(b^+ + 1) \pmod{4}$. Formally, one can regard q_d as being given by the formula

$$q_d(h) = \langle \mu(h)^d, [M_d] \rangle,$$

where M_d is the instanton moduli space of dimension $2d$ (depending on a choice of a Riemannian metric on X) and μ is a natural map from $H_2(X)$ to $H^2(M_d)$. Because M_d is non-compact in general, this pairing needs to be correctly interpreted before it can be regarded as well-defined: a recipe for evaluation was given in [3], subject to the constraint $2d > 3(b^+ + 1)$, and by various devices the construction has since been extended [6,11] so as to remove this restriction.

Although the polynomial invariants have been a powerful tool in enlarging our understanding of smooth 4-manifolds, their nature has remained mysterious. The first applications rested on a small number of rather general properties, such as the vanishing theorem for connected sums proved in [3]. Other results have arisen from calculations of particular values of q_d for carefully chosen 4-manifolds; the elliptic surfaces are a good example [1,5,12,13]. Until now however, there has remained only one 4-manifold, the $K3$ surface, for which the invariants q_d are entirely known

1991 *Mathematics Subject Classification*. Primary 57R57, 57N13, 57R55; Secondary 57R40, 14J60.

(apart, that is, from cases to which the vanishing theorem applies). The isolated values computed in other examples have remained a short but slowly growing list.

The results which we announce here are not in the line of further particular calculations. Rather, we prove that there is a family of universal linear relations which constrain the values of q_d , for a large class of 4-manifolds (the manifolds of *simple type*; see section 2). It turns out that the structure revealed by these relations is best expressed by combining all the polynomials q_d into one analytic function on $H_2(X, \mathbb{R})$: our main result (Theorem 1) then describes the structure of this analytic function in terms of a distinguished collection of 2-dimensional cohomology classes, which we call the *basic classes* on X . Theorem 2 shows that the basic classes determine a lower bound on the genus of embedded surfaces in the 4-manifold; the bound is reminiscent of the way in which the canonical class of a complex surface determines the genus of embedded complex curves.

Section 4 explains the origin of the universal relations and gives some pointers to the proof of the theorems. Our main tools are the moduli spaces of instantons with singularities in codimension 2, introduced in [9,10].

2. STATEMENT OF RESULTS

Before stating our results, we must introduce the simple type condition (see [8]). In addition to the 2-dimensional cohomology classes $\mu(h)$, the moduli spaces M_d also carry a 4-dimensional class ν , and one can define a larger class of invariants of X by evaluating products such as $\mu(h)^a \nu^b$ on the fundamental class $[M_d]$, whenever $a + 2b = d$; see [4] for example. We adopt the convention that ν is $-\frac{1}{4}p_1(\mathbb{E})$, where \mathbb{E} is the principal $SO(3)$ bundle associated to the base-point fibration. We say that X has simple type if the 4-dimensional class satisfies the relation

$$\langle \mu(h)^a \nu^{b+2}, [M_{d+4}] \rangle = 4 \langle \mu(h)^a \nu^b, [M_d] \rangle, \quad (1)$$

for all d and all $h \in H_2(X)$. This condition is known to hold for a large class of spaces, including the manifolds underlying the simply-connected elliptic surfaces, complete intersections and various branched covers, as well as many ‘fake’ 4-manifolds made from these by surgeries and gluings. This list probably contains all examples for which any calculations have yet been made, and it is not impossible that (1) is valid for all simply-connected 4-manifolds.

Whenever (1) holds, the evaluation of $\mu(h)^a \nu^b$ can be reduced either to the case $b = 0$, which gives the original q_d , or to the case $b = 1$. As a matter of notation, it is then convenient to incorporate the case $b = 1$ by introducing invariants q_d for all $d \equiv \frac{1}{2}(b^+ + 1) \pmod{2}$ (rather than just mod 4) by defining

$$2q_{d-2}(h) = \langle \mu(h)^{d-2} \nu, [M_d] \rangle.$$

If d is not equal to $\frac{1}{2}(b^+ + 1) \pmod{2}$, or if d is negative, we set $q_d = 0$. We then combine all the polynomials q_d into one analytic function or formal power series $q : H_2(X, \mathbb{R}) \rightarrow \mathbb{R}$ by defining

$$q(h) = \sum_d q_d(h)/d!$$

Our main result is summarized in the following two theorems.

Theorem 1. *If X is a simply-connected 4-manifold of simple type, then there exist finitely many cohomology classes $K_1, \dots, K_p \in H^2(X, \mathbb{Z})$ and non-zero rational numbers a_1, \dots, a_p such that*

$$q = \exp\left(\frac{Q}{2}\right) \sum_{s=1}^p a_s e^{K_s}$$

as analytic functions on $H_2(X, \mathbb{R})$. Here Q is the intersection form, regarded as a quadratic function. Each of the ‘basic classes’ K_s is an integral lift of $w_2(X)$.

Theorem 2. *Let X be again a simply-connected 4-manifold of simple type and let $\{K_s\}$ be the set of basic classes given by Theorem 1. If Σ is any smoothly embedded, essential connected surface in X with normal bundle of non-negative degree, then the genus of Σ satisfies the lower bound*

$$2g - 2 \geq \Sigma \cdot \Sigma + \max_s K_s \cdot \Sigma.$$

We make some remarks about these results. First of all, since q is always either an even or an odd function depending on the parity of $\frac{1}{2}(b^+ + 1)$, the non-zero basic classes come in pairs: if K is a basic class then so is $-K$, and the sum of exponentials which appears in Theorem 1 is a sum of hyperbolic cosines in the even case or hyperbolic sines in the odd case. It follows too that the function

$$J = \max_s K_s$$

which appears in Theorem 2 is non-negative and even; it can be thought of as defining a piece-wise linear semi-norm on $H_2(X)$, and the inequality

$$2g - 2 \geq Q + J$$

of the Theorem makes it reminiscent of the Thurston norm on the homology of a 3-manifold [14]. As we said in the Introduction, the inequality is also reminiscent of the adjunction formula for the genus of a smooth complex curve: we choose the notation K for the basic classes because we think of them as generalizations of the canonical class of a complex surface. Note that the function J can also be extracted from the asymptotics of q . As $h \rightarrow \infty$ in $H_2(X, \mathbb{R})$, we have

$$\log q(h) = Q(h)/2 + J(h) + O(1).$$

3. EXAMPLES

The first example is the $K3$ surface, whose invariants are known to be given by the formula

$$q_{2i} = \frac{(2i)!}{2^i i!} Q^i$$

for all even degrees $2i$. The analytic function q therefore has the expression

$$q = \exp\left(\frac{Q}{2}\right).$$

Thus the only basic class for $K3$ is the zero class. Consider next the complex surface X formed as a double cover of $\mathbb{C}\mathbb{P}^2$, branched along a smooth octic curve. The invariants of X are known to be polynomials in the canonical class K_X and the intersection form Q ; so the basic classes K must all be multiples of K_X – integer multiples in fact, since K_X is primitive. Because of the last clause of Theorem 1, only odd integer multiples can occur, as $w_2(X)$ is non-zero. Now X contains smooth complex curves Σ whose genera are given by the adjunction formula $2g - 2 = \Sigma \cdot \Sigma + K_X \cdot \Sigma$, so according to Theorem 2 we have

$$K(\Sigma) \leq K_X(\Sigma),$$

for all basic classes K . Putting these facts together we see that the only basic classes are $\pm K_X$. Since the function q is even in this case and since one coefficient of one polynomial was calculated in [4], we have enough information to conclude that

$$q = 2 \exp\left(\frac{Q}{2}\right) \cosh K_X. \quad (2)$$

This gives us the entire invariant for the complex surface X .

Elliptic surfaces provide other examples where Theorem 1 can be combined with previous calculations of particular coefficients [5] to yield a complete answer. For a simply connected minimal elliptic surface with no multiple fibres, the invariant can be shown to be

$$q = \exp\left(\frac{Q}{2}\right) (\sinh F)^{p_g-1}, \quad (3)$$

where p_g is the geometric genus (which should be positive) and F is the cohomology class dual to the generic elliptic fibre. The basic classes in this case are nF for integers n in the range $|n| \leq p_g - 1$ satisfying $n \equiv p_g - 1 \pmod{2}$. Note that $(p_g - 1)F$ is the canonical class of the surface.

The effect of forming a connected sum with $\overline{\mathbb{C}\mathbb{P}^2}$ in these examples can also be calculated. If X is $K3 \# \overline{\mathbb{C}\mathbb{P}^2}$ for example, we have

$$q = \exp\left(\frac{Q}{2}\right) \cosh E, \quad (4)$$

where E is dual to the generator of $\overline{\mathbb{C}\mathbb{P}^2}$ and Q denotes the intersection form of X (rather than the intersection form of $K3$). The basic classes are therefore $\pm E$. The formula has the same shape in examples (2) and (3) also: thus if $\hat{X} = X \# \overline{\mathbb{C}\mathbb{P}^2}$ where X is the double cover of $\mathbb{C}\mathbb{P}^2$ branched over a smooth octic, then the basic classes are $\pm K_X \pm E$ and the invariant is

$$q = 2 \exp\left(\frac{Q}{2}\right) \cosh K_X \cosh E.$$

4. STRUCTURE OF THE PROOF

It is convenient to introduce the function $C = \exp(-Q/2)q$ on $H_2(X, \mathbb{R})$ and to separate it into its homogeneous parts:

$$C(h) = \sum_d C_d(h)/d!$$

Thus C_d is a polynomial of degree d on $H_2(X, \mathbb{R})$, and it is related to q_d by

$$C_d = \sum_{i=0}^{\lfloor d/2 \rfloor} \frac{(-1)^i d!}{(d-2i)! i! 2^i} Q^i q_{d-2i}. \tag{5}$$

The content of Theorem 1 is that $C(h)$ is a linear combination of exponentials. If we focus on a particular class S in the positive cone in $H_2(X, \mathbb{Z})$, then the central thrust of the result is contained in the following proposition.

Proposition 3. *For any $S \in H_2(X, \mathbb{Z})$ with $Q(S)$ positive, there is an expression*

$$C_d(S) = \sum_{s \in \mathbb{Z}} \alpha_s s^d,$$

valid for all $d \geq 0$, with only finitely many non-zero terms. If S is represented by an embedded surface of genus g , then α_s is zero for $|s| > 2g - 2 - Q(S)$. Further, α_s is non-zero only when $s \equiv Q(S) \pmod{2}$.

We can view this as saying that the sequence $\{C_d(S)\}$ ($d \in \mathbb{N}$) satisfies a finite-order linear recurrence relation with integer roots. These recurrence relations are the universal relations mentioned in the introduction.

Let (X, Σ) be a pair consisting of an oriented 4-manifold of simple type and an oriented embedded surface. In [9] and [10] it was shown how to associate to (X, Σ) a family of polynomial invariants $q_{k,l}$, generalizing the ordinary polynomial invariants of X . Their definition followed the definition of the usual polynomials [3] but used moduli spaces of instantons with a specified singularity along Σ . The degree of $q_{k,l}$ is a function of k and l and the topology of the pair, and like the degree of q_d , its parity is constrained. We are interested only in the case that $q_{k,l}$ has degree zero, so for any given l , we define r_l to be $q_{k,l}$ if we can find k such that the degree is zero. Otherwise, we define r_l to be zero. Thus for each $l \in \mathbb{Z}$, we have an integer-valued invariant r_l for pairs (X, Σ) .

One of the main results of [10] is that if Σ has odd genus and positive square, and if we set $l_o = (g - 1)/2$, then the invariant r_{l_o} can be expressed in terms of the ordinary polynomials: we have

$$r_{l_o} = 2^g q_0. \tag{6}$$

To obtain the recurrence relation, we first generalize (6) to express r_{l_o-p} in terms of the q_d . If $p < 0$ then this invariant vanishes [10], and for $p > 0$ we establish a universal formula

$$r_{l_o-p} = A^{p,0} q_{2p}(\Sigma) + A^{p,2} q_{2p-2}(\Sigma) + \dots, \tag{7}$$

where $A^{p,2i}$ is a quantity depending on the genus and self-intersection number of Σ . The leading term $A^{p,0}$ is non-zero. This formula is not proved in complete generality, but it is shown to hold when the genus and self-intersection number of Σ are large compared to p .

The invariants r_l satisfy rather simple relations which are quite easy to prove [10,8]. In particular, if the homology class of Σ is divisible by 2, then we have

$$r_{l_0-p} = \pm r_{l_0-p'} \quad \text{where} \quad p + p' = \frac{1}{2}(2g - 2 - \Sigma \cdot \Sigma).$$

Using the formulae (7), we then obtain linear relations amongst the values of $q_d(\Sigma)$. When expressed in terms of the values of $C_d(\Sigma)$ using the formula (5), these linear relations take the form of a recurrence relation with integer roots. The degree of the recurrence relation depends on the genus of Σ on account of the relationship between p and p' . The argument only establishes the validity of the recurrence relation when d is small compared to the genus and self-intersection number, because the formula (7) is not proved for large p . So given a homology class S of positive square, we apply the argument to embedded surfaces Σ representing various large multiples of S to obtain eventually a recurrence relation on $C_d(S)$ valid for all d .

The proof is made more complicated by the fact that we are unable directly to calculate the coefficients $A^{p,2i}$ in the formula (7). Instead, we establish the nature of the resulting linear relations by an indirect argument based on known examples. The main input is our knowledge of the invariants for $K3$ and some coefficients for elliptic surfaces without multiple fibres [12]. An important stepping-stone is the proof of the formula (4) for the invariants of $K3\#\mathbb{C}P^2$. Details of the proof, as well as some more detailed applications, will appear in a later paper.

5. QUESTIONS

It is natural to ask whether the basic classes K_s can be shown to satisfy any other constraints relating perhaps to the homotopy type of X . The canonical class of a complex surface X satisfies

$$K_X^2 = 2\chi + 3\sigma$$

where χ is the Euler number and σ is the signature of X . In the few examples which we know, all the basic classes K_s satisfy this same constraint, so it is possible that it is a general property. There is little to go on at the moment.

One must also ask to what extent the inequality for the genus in Theorem 2 is sharp. The function J on $H_2(X, \mathbb{Z})$ which appears there is essentially the same as the function J which was defined in section 5(iii) of [8], at least on the positive cone. In the case that X is a complex surface whose invariants are polynomials in Q and K_X , one can show that $J(S) = |K_X \cdot S|$ by combining the results of [7] with the material of this paper. This means showing that K_X is one of the basic classes.

The question of whether Theorem 2 is sharp might be examined in connection with a relationship between J and the Thurston norm. Let Y be a 3-manifold with non-trivial homology $H_2(Y, \mathbb{R})$, and consider the flat connections over Y . If we use an $SO(3)$ bundle with suitably chosen Stiefel-Whitney class, we may arrange that there are no reducible connections, and under these circumstances it is possible

to define a Floer homology group $HF(Y)$. The construction of the polynomial invariants gives, in this situation, a linear map Φ from $H_2(Y, \mathbb{R})$ to the ring of endomorphisms of $HF(Y)$. All elements of the image of Φ commute [2], so they have simultaneous eigenvalues which can be regarded as defining r linear functions on $H_2(Y, \mathbb{R})$, where r is the rank of the Floer homology: these are close cousins of the basic classes in this paper. On the basis of the results we have described, one might speculate that the linear functions defined by the eigenvalues are integral, and that their supremum defines a semi-norm J_Y on $H_2(Y, \mathbb{R})$ which provides a lower bound for the Thurston norm x . It would then be interesting to know if there is any significant class of 3-manifolds for which x and J_Y can be shown to be equal.

The role of the simple type condition should also be clarified. If there are simply-connected 4-manifolds which are not of simple type, it is possible that the statement of Theorem 1 would need only minor modification to cover the general case. Perhaps the constant coefficients a_s which appear there should be replaced by polynomial functions.

We close with a conjecture about the invariants of elliptic surfaces, rather less speculative than the remarks above. Let X be a regular elliptic surface with geometric genus $p_g \geq 1$ and r multiple fibres of multiplicities m_1, \dots, m_r . We want X to be simply connected, which means that we must have $r \leq 2$, and in the case $r = 2$ the two multiplicities must be coprime. Then we conjecture that

$$q = \exp\left(\frac{Q}{2}\right) \frac{(\sinh F)^{p_g-1+r}}{\prod_i \sinh(F/m_i)},$$

where F is the class dual to the generic fibre. For $r = 0$ this is the formula given above in section 3. For $r = 1$ the formula is correct in a few small cases such as when $p_g = 1$ and $m_1 \leq 4$, or $p_g = 2$ or 3 and $m_1 = 2$. For $r = 2$ and all p_g , the formula is in agreement with the known expressions for the first two non-zero coefficients of q_d in its expansion as a polynomial in Q and F [12]. The results of this paper will probably extend without change to the non-simply connected case, as long as $H_1(X, \mathbb{R}) = 0$, so the restriction $r \leq 2$ should not be regarded as essential.

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