ON SOME OF THE RESIDUAL PROPERTIES OF FINITELY GENERATED NILPOTENT GROUPS

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Abstract. In recent years, the RFRS condition has been used to analyze virtual fibering in 3-manifold topology. Agol's work shows that any 3-manifold with zero Euler characteristic satisfying the RFRS condition on its fundamental group virtually fibers over the circle. In this note we will show that a finitely generated nilpotent group is either virtually abelian or is not virtually RFRS. As a corollary, we deduce that any RFRS group cannot contain a nonabelian torsion-free nilpotent group. This result also illustrates some of the interplay between residual torsion-free nilpotence and the RFRS condition. We close with a topological approach to the residual torsion–free nilpotence of free, surface and graph groups.

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1. Introduction

Let Γ be a finite undirected graph. By this we mean that Γ consists of a finite set V of vertices and a finite set $E \subset (V \times V \setminus \Delta)/\mathbb{Z}/2\mathbb{Z}$ of edges. We consider edges as unordered pairs of vertices so that the edges are undirected, and we assume that there is at most one edge connecting any two vertices. Removing the diagonal from $V \times V$ prevents any edges looping at a vertex. We define the **graph group** $G(\Gamma)$ associated to Γ by the presentation

$$\langle V \mid [v_1, v_2] \text{ if } \{v_1, v_2\} \in E\rangle.$$

The graph group $G(\Gamma)$ is often called a right-angled Artin group (RAAG).

Recall that a group G is called **residually finite rationally solvable** or **RFRS** if there exists an exhaustive filtration of normal (in G) finite index subgroups of G

$$G = G_0 > G_1 > G_2 > \cdots$$

such that $G_{i+1} > ker\{G_i \to G_i^{ab} \otimes \mathbb{Q}\}$. Agol proves:

Theorem 1.1 ([A]). *Let M be a 3-manifold with $\chi(M) = 0$, and suppose that $\pi_1(M)$ is RFRS. Let $\phi \in H^1(M, \mathbb{Z})$ be a non-fibered cohomology class. Then there is a finite cover $M' \to M$ such that the pullback of ϕ to $H^1(M', \mathbb{Z})$ of ϕ lies in the cone over the boundary of a fibered face of $H^1(M', \mathbb{Z})$.*

He then deduces:

Corollary 1.2. *Let M be a 3-manifold with $\chi(M) = 0$ and suppose that $\pi_1(M)$ virtually embeds in a graph group. Then M virtually fibers over the circle.*

The interest in graph groups as they relate to virtual fibering is partially due to the following theorem of Wise (cf. the work in [HW]):

Theorem 1.3 ([W1],[W2]). *Let G be a word-hyperbolic group with a quasi-convex hierarchy. Then G has a finite index subgroup G' which embeds in a graph group R.*

Wise uses Agol's theorem to deduce:

Corollary 1.4. *A Haken hyperbolic* 3*-manifold of finite volume is virtually fibered.*

In [A], Agol provides a proof that right-angled reflection groups are virtually RFRS. On the other hand:

Theorem 1.5 (Droms ([D]), Duchamp and Thibon ([DT]), A. Vijayan ([C1]), and others). *Let G be a graph group. Then G is residually torsion-free nilpotent.*

Residual torsion-free nilpotence and the RFRS condition are independent since it is known that there exist residually torsion-free nilpotent groups which are not even virtually RFRS (see [K]). Examples of such groups are compact 3-manifolds with Nil geometry and nontrivial circle bundles with trivial monodromy over a closed surface.

To further illustrate the incompatibility of RFRS and residual torsion-free nilpotence, we have the following result:

Theorem 1.6. *Let N be a finitely generated nilpotent group. Then N is either virtually abelian or N is not virtually RFRS.*

The RFRS condition was originally developed to answer questions about 3-manifold groups, and nilpotent groups never arise as fundamental groups of finite volume hyperbolic 3-manifolds. Theorem 1.6 is not so much a result with applicability to 3-manifold theory but rather one which might be of independent interest.

Since the RFRS condition is inherited by subgroups, we deduce:

Corollary 1.7. *Let G be a virtually RFRS group. Then G does not contain a torsion-free nilpotent group.*

In particular it follows that graph groups do not contain torsion-free nilpotent groups, though this fact can be deduced from the Tits alternative for graph groups.

The last section of the paper is devoted to a topological perspective on residual torsion–free nilpotence, and we discuss in particular the cases of free, surface and graph groups.

2. Acknowledgments

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3. Proof of the main result

In [K], the author proves that the Heisenberg group is not RFRS. An easy modification of the proof shows that the Heisenberg group is in fact not virtually RFRS. The author also proves that a nontrivial circle bundle with trivial monodromy over a closed orientable surface has a residually torsion-free nilpotent fundamental group, but remarks that it cannot be RFRS by Agol's theorem. The author then proves directly that the fundamental group of such a manifold cannot be RFRS. We now adapt this discussion to general finitely generated nilpotent groups.

Lemma 3.1. *Let N be a finitely generated torsion-free nilpotent group which is virtually abelian. Then N is abelian.*

Proof. Let $N' < N$ be a finite index normal subgroup of N which is abelian. Since N' is finitely generated, it is isomorphic to $\mathbb{Z}^n$, and the quotient N/N' is some finite group F. The conjugation action of N on N' factors through F, since N' is abelian and therefore acts trivially. We thus obtain a representation $F \to GL_n(\mathbb{Z})$, whose image is nontrivial since we may assume N is nonabelian. But the image of any element of F in $GL_n(\mathbb{Z})$ is semisimple since F is finite (it can be diagonalized over $\mathbb{C}$). It follows that the action of F on N' cannot be unipotent. In particular, let $f \in F$ have image $A_f \in GL_n(\mathbb{Z})$. Then $A_f - I$ is not nilpotent, so that $(A_f - I)^M(\mathbb{Z}^n) \neq \{0\}$ for all $M > 0$. It follows that N could not have been nilpotent. □

What the proof of Lemma 3.1 shows is that if A is a torsion-free abelian normal subgroup of a nilpotent group, then the conjugation action of N on A factors through N/A and must act as a unipotent group of automorphisms of A, which is not possible. If T is a non-unipotent matrix, then putting T in Jordan canonical form (over $\mathbb{C}$) we see that $(T-I)^n$ has positive rank (even over $\mathbb{Q}$), no matter how large n gets.

Recall that finitely generated nilpotent groups have a well-defined notion of **rank** (cf. [R]). Precisely, let N be finitely generated and nilpotent. Suppose that

$$N = N_1^1 > N_2^1 > \cdots > N_i^1 > N_{i+1}^1 = \{1\}$$

and

$$N = N_1^2 > N_2^2 > \cdots > N_j^2 > N_{j+1}^2 = \{1\}$$

are two normal filtrations of N such that N_i^1 and N_j^2 are abelian and each successive quotient is abelian. Then the rank of N, written $\operatorname{rk} N$, is given by

$$\sum_{k=1}^{i} \operatorname{rk} N_k^1/N_{k+1}^1 = \sum_{k=1}^{j} \operatorname{rk} N_k^2/N_{k+1}^2.$$

Such filtrations always exist, so that the rank is a well-defined nonnegative integer. Notice that by definition, if

$$1 \to N' \to N \to N'' \to 1$$

is a short exact sequence of nilpotent groups, where N is finitely generated, then $\operatorname{rk} N = \operatorname{rk} N' + \operatorname{rk} N''$.

Let N be a torsion-free nonabelian nilpotent group with center Z. It is standard that Z is again finitely generated (see [R]). Suppose that the extension

$$1 \to Z \to N \to N/Z \to 1$$

does not split. Let N' be a finite index subgroup of N. N' is still clearly torsion-free and nilpotent. It is conceivable that the extension

$$1 \to Z \cap N' \to N' \to N'/(Z \cap N') \to 1$$

does split. However, as a consequence of Lemma 3.1 and by induction on the rank of N, we see that the extension

$$1 \to Z(N') \to N' \to N'/Z(N') \to 1$$

does not split.

Lemma 3.2. *Let N be a finitely generated torsion-free nilpotent group and let $\{1\} \neq Z < Z(N)$ be a central subgroup. Suppose that the extension*

$$1 \to Z \to N \to N/Z \to 1$$

is not a split extension. Then either the restriction of $N \to N^{ab}$ to Z is not injective or N virtually splits as a nontrivial direct product.

Proof. By [E], we may suppose that N/Z is torsion-free. Note that it is not possible for N to be abelian, for if it was then the extension would automatically split. Suppose that the abelianization map restricted to Z is injective. Then we see that Z virtually splits off as a direct summand of N^{ab}, so that $N^{ab} \cong A \times Z'$ for some abelian group A and a torsion-free abelian group Z' which contains the image of Z with finite index. Let K be the kernel of the map $N \to Z'$. Then K fits into a (not necessarily central) extension of the form

$$1 \to K \to N \to A \times Z' \to 1.$$

This extension splits partially, since Z is a subgroup of N. Since Z is central, Z and K normalize each other in N. Since Z maps injectively to Z', $Z \cap K = \{1\}$. It follows that Z and K together generate a subgroup of N which is isomorphic to $K \times Z$. Since $N/(K \times Z) \cong Z'/Z$, the conclusion follows. □

Corollary 3.3. *Let N be a finitely generated torsion-free nilpotent group, and suppose that $Z = Z(N)$ maps injectively to N^{ab}. Then N is abelian.*

Proof. By Lemma 3.2, N virtually splits as a direct product of the form $K \times Z$. Since K is nilpotent, it has a center which is clearly a subgroup of N and is hence torsion-free, abelian, and finitely generated. Since Z is central, $K \times Z$ acts trivially on the center of K. Since Z is precisely the center of N, $X = Z(K)$ is not central in N. It follows that $\Gamma = N/(K \times Z)$ acts nontrivially on X. As $X \cong \mathbb{Z}^n$, the conjugation action of Γ is given by a representation $\Gamma \to GL_n(\mathbb{Z})$. Since Γ is a finite group (it is abelian as an easy consequence of the proof of Lemma 3.2), Γ acts by nontrivial semisimple automorphisms (they can in fact be diagonalized over $\mathbb{C}$). It follows that the conjugation action of Γ is not unipotent, which violates the nilpotence of N. It follows that if K is nontrivial then Z is not the whole center of N. It follows that N is virtually abelian and hence abelian. □

Proof of Theorem 1.6. Since N is polycyclic, we see that N is virtually torsion-free by [R]. So, we may assume that N is a nonabelian torsion-free nilpotent group. By the corollary to Lemma 3.1, it suffices to show that N is not RFRS. Let $Z = Z(N)$ and let $\{G_i\}$ be any filtration of N which witnesses the claim that N is RFRS, with $G_0 = N$. By Corollary 3.3, Z does not map injectively into N^{ab}. There is an element $z \in Z$ such that $H = \langle z \rangle$ is infinite cyclic and the canonical map $H \to N^{ab}$ is not injective. The fact that $H \to N^{ab}$ is not injective implies that H is contained in $ker\{N \to N^{ab} \otimes \mathbb{Q}\}$, so that H is contained in G_1.

We may assume that $H < G_i$ for some $i \geq 1$. Since G_i has finite index in N, we may assume that G_i is not abelian. Suppose that H maps injectively to G_i^{ab}. Then by Lemma 3.2, G_i virtually splits as a nontrivial direct product. By Corollary 3.3, there is another central cyclic subgroup H' of G_i which does not map injectively to G_i^{ab} and hence must be contained in G_{i+1}.

If

$$\bigcap_i G_i = \{1\},$$

then eventually each element z of the center of N must be contained in $G_i \setminus G_{i+1}$ for some index i. If $i = i(z)$ is such an index then G_i virtually splits as $K \times \mathbb{Z}$, where the $\mathbb{Z}$ factor generated by some power of z. Note that $\operatorname{rk} K + 1 \leq \operatorname{rk} N$. Since K is nilpotent, it also has a center. By induction on rank, N is virtually abelian. The conclusion follows by Lemma 3.1. □

For completeness, we include the following observation, which is implicit in the work of Agol and others:

Lemma 3.4. *Let G be a group which is RFRS and $H < G$ a subgroup. Then H is RFRS.*

Proof. Let $\{G_i\}$ be a filtration of G which witnesses the fact that G is RFRS. Clearly $\{H_i\} = \{G_i \cap H\}$ is a filtration on H. Since there is a map $H_i^{ab} \to G_i^{ab}$ given by the inclusion, if $h \in ker\{H_i \to H_i^{ab} \otimes \mathbb{Q}\}$ then h maps to a torsion element in G_i^{ab} and is therefore contained in G_{i+1}. It follows that h is contained in H_{i+1}. Conversely, if $h \in G_i \setminus G_{i+1}$ then h maps to a non-torsion element of G_i^{ab} and must therefore map to a non-torsion element of H_i^{ab}. □

It follows that if G is a virtually RFRS group then G cannot contain a nonabelian torsion-free nilpotent group.

4. Topological perspectives on residual torsion–free nilpotence

We will close with a proof of the residual torsion–free nilpotence of free groups, surface groups and graph groups using a topological method which, to the author's knowledge, was previously unknown. The proof will combine residually p–methods together with an algebraic condition which relates residually p groups and residually torsion–free groups.

The algebraic lemma we will need is as follows. Let G be a finitely generated group, $g \in G$, and let $\mathcal{P}$ be a set of primes. We say that $g \in G$ is $\mathcal{P}$–**good** if there is an $N = N(g)$ such that g survives in a p–group quotient of G of nilpotence degree no more than N for each $p \in \mathcal{P}$. The following can also be found in [K]:

Lemma 4.1. *Let G be a finitely generated group. The following are equivalent.*

1. *G is residually torsion–free nilpotent.*
2. *There is an infinite set of prime $\mathcal{P}$ such that each nontrivial $g \in G$ is $\mathcal{P}$–good.*
3. *Each nontrivial $g \in G$ is $\mathcal{P}$–good with respect to the set of all primes.*

Proof. Suppose G is residually torsion–free nilpotent, and let $1 \neq g \in G$. Let N be a torsion–free nilpotent quotient of G where g survives. By Mal'cev's Theorem (see [R]), we may embed N as a group U of integral unipotent matrices in some finite dimensional general linear group. Let p be any prime. Reducing the entries of U modulo p^n results in a finite p–group U_{p^n} whose nilpotence degree is no larger than that of N. Choosing n sufficiently large, we see that 1 implies 3. 3 implies 2 trivially. To see that 2 implies 1, let $1 \neq g \in G$ and for each $P \in \mathcal{P}$, let P_g be an at most degree N nilpotent P–group in which g survives. Write

$$M = \prod_{P \in \mathcal{P}} P_g,$$

and let N be the image of G in M. Notice that M is at most N–step nilpotent, and N is a finitely generated subgroup of M in which g has infinite order. The torsion elements of N form a normal subgroup of N (see [E] or [R]), so there is a torsion–free quotient N' of N in which g survives. In particular, G is residually torsion–free nilpotent. □

A related characterization of torsion–free nilpotence is the following:

Lemma 4.2. *A finitely generated group is residually nilpotent if any only if for each nontrivial $g \in G$ there is a prime p and a sequence of p–group quotients $\{P_m\}_{m \in \mathbb{N}}$ of G of bounded nilpotence degree such that the minimal order of a nontrivial element in P_m is p^m and such that g survives in P_m.*

Proof. Again, consider the product

$$P = \prod_m P_m$$

and the image of g in P. Obviously g is not sent to a torsion element and the image of G is finitely generated, so that G is residually torsion–free nilpotent as in the proof of Lemma 4.1. Conversely, we again appeal to Mal'cev's Theorem and reduce modulo powers of p. □

We now give the relevant topological argument:

Lemma 4.3. *Let G be a finitely generated free group or a surface group and let p be a prime. Then G is residually p.*

Proof. Suppose first that G is free. Identify G with the fundamental group of a finite wedge of circles X, which we endow with the graph metric. We let $X = X_0$ and we build a tower of finite covering spaces by taking X_{i+1} to be the cover of X_i corresponding to $H_1(X_i, \mathbb{Z}/p\mathbb{Z})$. Note that $X_i \to X_0$ is a normal covering space of p–power degree.

Let Γ be a finite graph. Let γ be a loop in Γ, which we view as a path with the same initial and terminal vertex. We say that γ is a simple loop in Γ if it visits each vertex other than the initial one at most once. Note that γ represents a nontrivial homology class modulo p. Note also that in any finite graph with the graph metric, any minimal length loop is always simple. It follows that any loop of length k in X_0 does not lift to X_k.

For surface groups there is an additional complication, which is that short loops may be homologically trivial. Let $G = \pi_1(\Sigma)$, and we will choose a discrete, cocompact embedding $G \to PSL_2(\mathbb{R})$ so that Σ is then a quotient of the hyperbolic plane $\mathbb{H}^2$. Each essential homotopy class of curves in Σ is now represented by a hyperbolic geodesic. We again let $\Sigma = X_0$ and we construct the tower of p–power covering spaces as before.

Note that with the hyperbolic metric, any shortest closed geodesic is still simple. If $\gamma \subset X_i$ is a shortest length closed geodesic, we record the homology class of γ. If it is nontrivial then γ is nonseparating and hence is nontrivial in $H_1(X_i, \mathbb{Z}/p\mathbb{Z})$. Otherwise, γ is separating and each lift of γ to X_{i+1} is nonseparating, so that γ does not lift to a closed loop in either X_{i+1} or X_{i+2}. The length spectrum of geodesics in X_0 is discrete, so that as i tends to infinity, the length of the shortest loop in X_0 which lifts to X_i also tends to infinity. □

Let $\Gamma^p_{n,m}$ be the universal n–step nilpotent p–group quotient of G of exponent p^m. Write $K^p_{n,m}$ for the kernel of the projection $G \to \Gamma^p_{n,m}$. Lemma 4.3 shows that for each prime p,

$$\bigcap_{n,m\geq 0} K^p_{n,m} = \{1\}.$$

Note that $K^p_{n,m}$ is characteristic in G but not necessarily in $K^p_{n-1,m}$.

There is a general principle in the theory of free groups that if one considers a sequence $\{K_n\}$ of properly nested subgroups of a finitely generated free group F with a free generating set S with K_{n+1} characteristic in K_n, then the shortest word in K_n with respect to S has length at least n (see [LS]). Unfortunately this result cannot be applied here by the observation in the previous paragraph.

In order to apply Lemma 4.1, we must do a little more work. Consider the setup of Lemma 4.3. Let H^p_n be the deck group of X_n over X_0 with respect to the prime p. Notice that the derived series of H^p_n has length at most n. If p and q are different primes, the nilpotence degrees of H^p_n and H^q_n might be wildly different.

Proposition 4.4. *Let $g \in G$ be nontrivial and let p be a prime. Then for each prime p, g satisfies the hypotheses of Lemma 4.2. In particular, G is residually torsion–free nilpotent.*

Proof. Let k be the length of g, which is to say either the word length in the free group with respect to a free generating set, or in the case of a surface group we say g has length k if the length of a geodesic representative of g is the k^{th} shortest possible length of a closed geodesic in a fixed hyperbolic metric. By Lemma 4.3, g survives in the quotient H^p_{2k} for each prime p. Let n be the nilpotence degree of H^p_{2k}.

We can repeat the construction in the proof of Lemma 4.3 by replacing the homology modulo p by the homology modulo p^m. Denote the deck group which is obtained from a tower of k spaces by $H^p_{2k,m}$. It is easy to check that each element of $H^p_{2k,m}$ will have order at least p^m, and it is obvious that g survives in $H^p_{2k,m}$. The nilpotence degree of $H^p_{2k,m}$ might be much larger than n. On the other hand, $H^p_{2k,m}$ has a universal n–step nilpotent quotient N, and we claim that g survives in N. This is easy to check by induction on n. The conclusion of the proposition follows by Lemma 4.2. ☐

It is well–known and mentioned in the introduction that graph groups are also residually torsion–free nilpotent. A similar argument to the one in this section could be used to provide a topological proof of this fact.

Proposition 4.5. *Graph groups are residually torsion–free nilpotent.*

Sketch of a proof. We let X_0 be the Salvetti complex of the graph group G. It is well–known that X_0 is a CAT(0) metric space. For more details on this construction, the reader might consult Charney's article [C2]. We construct for each prime the tower of covers $\{X_i\}$ as in the proof of Lemma 4.3. We now consider simple geodesic loops in each X_i, which are not defined by which vertices they visit but rather which walls they intersect. A closed geodesic is simple if it intersects any given wall at most once. If a loop γ intersects a particular nonseparating wall exactly once then this wall is nonseparating and γ represents a nontrivial homology class modulo each prime. On the other hand, minimal length geodesics among all homotopy classes in some X_i are simple. Thus we can obtain that G is residually p for each prime p. To obtain residual torsion–free nilpotence, we mimic the proof of Proposition 4.4. ☐

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