# RESIDUAL PROPERTIES OF FIBERED AND HYPERBOLIC 3–MANIFOLDS

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Abstract. Let $p$ be a prime. In this paper, we classify the geometric 3–manifolds whose fundamental groups are virtually residually $p$. Let $M = M^3$ be a virtually fibered 3-manifold. We prove that $\pi_1(M)$ is always virtually residually $p$. Using recent work of Wise, we prove that every hyperbolic 3–manifold is either closed or virtually fibered and hence has a virtually residually $p$ fundamental group. We then investigate some conditions under which the fundamental group of $M$ is virtually residually torsion–free nilpotent.

January 17, 2011

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## 1. Introduction and statement of results

Let $M$ be a compact orientable 3–manifold with torus boundary. The following is a recent result of Perelman on a conjecture of Thurston (cf. [T2], [P1], [P2], [P3]):

**Proposition 1.1** (Geometrization theorem, cf. Thurston and Perelman)**.** *Every oriented irreducible closed 3–manifold with torus boundary can be cut along a finite collection of incompressible tori so that the resulting pieces are "geometric", in the sense that they admit finite volume geometric structures modeled on one of the eight model geometries.*

Recall that these geometries are $S^3$, $S^2\times\mathbb{R}$, $\mathbb{R}^3$, Nil, Sol, $\widetilde{PSL_2}(\mathbb{R})$, $\mathbb{H}^2\times\mathbb{R}$, and $\mathbb{H}^3$. The purpose of this paper is to develop some theory of pro–$p$ and pro–nilpotent groups which will allow us to understand the group theoretic

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1991 *Mathematics Subject Classification.* Primary 20E26; Secondary 57N10, 57M10.

*Key words and phrases.* Fibered 3–manifold, residually nilpotent, hyperbolic manifold.

properties of the fundamental groups of compact manifolds modeled on these geometries.

Throughout this paper, fix a prime $p$. Let $M = M^3$ be a fibered 3–manifold. By this we shall consistently mean that there is a fibration

$$\Sigma \to M \to S^1,$$

where $\Sigma$ is a topological surface. Once we are given a fibered 3–manifold, we will fix a fibration over the circle once and for all. By convention, $\Sigma$ will always be a surface of genus $g \geq 1$ with finitely many punctures. We will denote the monodromy of the fibration by $\psi$. When it is necessary to prevent confusion, we will write $M_\psi$ for the fibered manifold with monodromy $\psi$. We get a corresponding short exact sequence of groups

$$1 \to \pi_1(\Sigma) \to \pi_1(M) \to \mathbb{Z} \to 1,$$

so that $\pi_1(M)$ splits as a semidirect product of $\pi_1(\Sigma)$ and a copy of $\mathbb{Z}$. In notation which will persist throughout this paper, the copy of $\mathbb{Z}$ is generated by a symbol $t$ which we call the **stable letter**.

If $\mathcal{K}$ is a class of groups, we say that a group $G$ is **residually** $\mathcal{K}$ if every nonidentity element of $G$ persists in a $\mathcal{K}$–quotient of $G$. In this paper, $\mathcal{K}$ will usually denote the class of finite $p$–groups or the class of torsion–free nilpotent groups. We will try to understand the failure or success of $\pi_1(M_\psi)$ to be residually $p$ from the data of the mapping class $\psi$ and how it acts on the homology of the fiber. Let $\overline{\mathbb{F}}_p$ denote an algebraic closure of the finite field with $p$ elements. There is a natural map

$$\mathbb{Z}/p\mathbb{Z} \to \overline{\mathbb{F}}_p.$$

If $A$ is an invertible matrix over $\mathbb{Z}/p\mathbb{Z}$, we say that $A$ is **unipotent** or **acts unipotently** if its only eigenvalue over $\overline{\mathbb{F}}_p$ is 1. Equivalently and without reference to any algebraically closed fields, the characteristic polynomial of $A$ should be $(x-1)^n$ for some $n$. We prove:

**Theorem 1.2.** *Let $G = \pi_1(M)$, where $M$ is fibered with fiber $\Sigma$ and monodromy $\psi$. If $\psi$ acts unipotently on $H_1(\Sigma, \mathbb{Z}/p\mathbb{Z})$, then $G$ is residually $p$. In particular, if $\psi$ is arbitrary then $G$ is virtually residually $p$. If $\Sigma$ is a torus, then $G$ is residually $p$ if and only if $p$ divides the determinant of $A - I$, where $A \in SL_2(\mathbb{Z})$ is the matrix which expresses the action of $\psi$ on the homology of the torus.*

We shall show that it is generally not necessary for $\psi$ to act unipotently on the modulo $p$ homology of $\Sigma$ for $G$ to be residually $p$.

Combining Theorem 1.2 with some elementary observations shows that nearly all geometric 3–manifolds have virtually residually $p$ fundamental groups. Indeed, $S^3$–manifolds have finite fundamental groups, $(S^2 \times \mathbb{R})$–manifolds have virtually cyclic fundamental groups, $\mathbb{R}^3$–manifolds have virtually abelian fundamental groups, and Nil, Sol, and $(\mathbb{H}^2 \times \mathbb{R})$–manifolds all virtually fiber over the circle. It is a fundamental open problem to show that all hyperbolic 3–manifolds virtually fiber over the circle. Wise has recently announced a proof that each Haken hyperbolic 3–manifold is virtually fibered over the circle (see [W1], [W2], [WK]). We prove the following result about hyperbolic 3–manifold:

**Theorem 1.3.** *Suppose each Haken hyperbolic* $3$*–manifold virtually fibers over the circle, and let* $G < PSL_2(\mathbb{C})$ *be a lattice. Then* $G$ *is virtually residually* $p$*.*

Whereas general theory of finitely generated linear groups gives us the conclusion of Theorem 1.3 for all but finitely many primes, it seems that the full conclusion is particular to hyperbolic orbifolds. Combining Theorem 1.3 with the work of Wise, we get a positive answer to a question of Lubotzky: for each prime $p$, all hyperbolic 3–orbifold groups are virtually residually $p$.

The remaining geometry to be considered is $\widetilde{PSL_2}(\mathbb{R})$–geometry. We can show that manifolds modeled on this geometry are always virtually residually $p$, but we will actually show something stronger and in the process give a partial answer to a question due to H. Wilton in [Wi].

In [Wi], Wilton classifies all compact 3–manifolds whose fundamental groups are residually free groups. Any such manifold has a residually $p$ fundamental group for each $p$ since free groups are residually $p$. However, most of the manifolds considered in this paper are not on the list of residually free 3–manifolds. For instance, he shows that no finite volume hyperbolic 3–manifold has a residually free fundamental group. He asks in that paper which compact 3–manifolds have residually torsion–free nilpotent fundamental groups. Note that any such group is residually $p$ for all $p$ by the work of Mal'cev. As for Wilton's question:

**Theorem 1.4.** *Suppose* $M$ *is a fibered* $3$*–manifold with fiber* $\Sigma$ *and suppose that the monodromy of the fibration acts unipotently on* $H_1(\Sigma,\mathbb{Z})$*. Then* $\pi_1(M)$ *is residually torsion–free nilpotent.*

In particular, any manifold built as a fiber bundle with monodromy in the Torelli group has a residually torsion–free nilpotent fundamental group. We also prove the following:

**Theorem 1.5.** *Let* $M$ *be a manifold modeled on* $\widetilde{PSL_2(\mathbb{R})}$*–geometry. Then* $G = \pi_1(M)$ *is virtually residually torsion–free nilpotent.*

From the proof of Theorem 1.5, we will obtain:

**Corollary 1.6.** *Let* $M$ *be a manifold modeled on* $\widetilde{PSL_2(\mathbb{R})}$*–geometry. Then* $G = \pi_1(M)$ *is linear.*

This fact was known and is mentioned in de la Harpe's book [dlH], but the proof we give here is knew as far as the author knows.

This paper stems from an attempt to understand the action of a mapping class on the homology of a surface, or more generally on the homology of a finite cover. Of particular interest was how mapping classes in the Torelli group $I(\Sigma)$ act on finite covers. The hope was initially that the residual properties of the resulting mapping torus would give some insight, though it did not. We can show however that if a particular mapping class $\psi$ gives rise to a residually $p$ fibered 3–manifold, then the residually $p$ property of the fundamental group depends only on the coset of $\psi$ under a particular representation, namely the modulo $p$ homology representation of the mapping class group:

**Theorem 1.7.** *Suppose $G_\psi = \pi_1(M_\psi)$ is residually $p$ and let $\phi$ be in the kernel of the modulo $p$ homology representation*

$$\mathrm{Mod}(\Sigma) \to GL_n(\mathbb{Z}/p\mathbb{Z}).$$

*Then $G_{\psi\circ\phi}$ is residually $p$.*

## 2. Acknowledgements

This work has benefitted from conversations with I. Agol, N. Avni, J. Behrstock, T. Church, B. Farb, S. Friedl, A. Lubotzky, C. McMullen and B. McReynolds. The idea for this paper arose from a conversation with W. Cavendish and R. Laverdiere. The author has been informed that in [AF1] and [AF2], M. Aschenbrenner and S. Friedl have independently obtained many of the results in this paper. The author was supported by an NSF Graduate Research Fellowship for part of this research. The author is indebted to the referee for numerous helpful comments and suggestions.

## 3. Tools for analyzing $p$–groups

We will record some tools from the theory of $p$–groups which will be at the heart of many of the arguments in this paper. We give either proofs or appropriate citations. The following is a standard fact about $p$–groups:

**Lemma 3.1.** *Let $G$ be a group and $K_1, K_2$ two $p$-power index normal subgroups. Then $G/(K_1 \cap K_2)$ is a $p$-group.*

*Proof.* Since $K_1$ and $K_2$ normalize each other, we can form the group $H = K_1K_2$. Evidently $H$ is normal in $G$ and has $p$-power index. By the second isomorphism theorem for groups, we have that $H/K_1 \cong K_2/(K_1 \cap K_2)$. The left hand side is obviously a $p$-group, and we also have $G > K_2 > (K_1 \cap K_2)$, whence $G/(K_1 \cap K_2)$ is a $p$-group. □

Without the assumption that at least $K_1$ is normal, the intersection $K_1 \cap K_2$ may fail to have $p$-power index as can be seen by considering $A_4 < A_5$ and noting that the intersection of all conjugates of $A_4$ is trivial.

If $G$ is a finitely generated group, then it has a maximal subgroup. Recall that the intersection of all maximal subgroups of $G$ is called the **Frattini subgroup** of $G$, and is denoted $\varphi(P)$. The Frattini subgroup of any group is obviously characteristic.

**Lemma 3.2.** *Let $P$ be a $p$-group. Then $P/\varphi(P)$ is the largest elementary abelian quotient of $P$.*

*Proof.* [KuSt], page 106. □

The following Lemma is important for proving a partial converse to Theorem 1.2:

**Lemma 3.3.** *Let $N$ be a nilpotent group with cyclic abelianization. Then $N$ is cyclic.*

*Proof.* Write $A$ for the abelianization of $N$. There is a nonabelian nilpotent quotient $\Gamma$ of $N$ which fits into a nontrivial central extension of the form

$$1 \to Z \to \Gamma \to A,$$

where $Z$ is an abelian group. This extension is classified by its Euler class, which sits in $H^2(A, Z)$. Since $A$ is cyclic it has cohomological dimension 1, so that the Euler class must vanish. □

For any group $G$, we write $\gamma_i(G)$ for the $i^{th}$ term of the **lower central series** of $G$. This is to say $\gamma_1(G) = G$ and

$$\gamma_{i+1}(G) = [\gamma_i(G), G].$$

We write $\mathcal{L}_i(G)$ for the quotient $\gamma_i(G)/\gamma_{i+1}(G)$. For a prime $p$, we write $\mathcal{L}_i(G, p)$ for $\mathcal{L}_i(G) \otimes \mathbb{Z}/p\mathbb{Z}$.

Let $\alpha \in \mathrm{Aut}(G)$ of finite order $\mu$. We will write $G_\alpha$ for the semidirect product

$$1 \to G \to G_\alpha \to \mathbb{Z}/\mu\mathbb{Z} \to 1,$$

where the stable letter $t$ which generates $\mathbb{Z}/\mu\mathbb{Z}$ acts on $G$ by $\alpha$ via conjugation.

The following result is a generalization of the fact that if $\alpha$ acts trivially on $H_1(G, \mathbb{Z})$ then it acts trivially on each quotient $\mathcal{L}_i(G)$.

**Lemma 3.4.** *Let $P$ be a $p$–group. Suppose that $\alpha$ is unipotent as a matrix acting on $\mathcal{L}_i(P, p)$ for all $i$. Then $P_\alpha$ is a $p$–group.*

*Proof.* Write $A$ for the associated matrix for the action of $\alpha$ on $P/\varphi(P)$. Forming a commutator $[t, v]$ for $v \in P/\varphi(P)$ is the same as applying the matrix $A - I$, in the sense that $[t, v] = (A - I)v$. The matrix $A - I$ is nilpotent since $A$ acts unipotently on $P/\varphi(P)$. Similarly, forming the commutator $[t^j, v]$ is the same as applying $A^j - I$, which is also nilpotent. Furthermore, the operators $A^i - I$ all commute with each other. So, any nested commutator can be written as

$$\left(\prod_{i=1}^{k}(A^i - I)^{n_i}\right) v$$

for some $v \in P/\varphi(P)$. By the pigeonhole principle any sufficiently long commutator will be trivial (since $A$ has finite order as an automorphism of $P/\varphi(P)$), so that the image of all sufficiently long commutators is in the kernel of the map

$$\mathcal{L}_i(P) \to \mathcal{L}_i(P, p).$$

Notice that each $\mathcal{L}_j(P)$ is a quotients of some free abelian group $\mathbb{Z}^n$, so we may lift the action of $\alpha$ (which is given by a matrix $A(j)$) to an integral matrix. We have that each sufficiently long product of the form

$$\left(\prod_{i=1}^{k}(A(j)^i - I)^{n_i}\right)$$

is an operator which sends $\mathbb{Z}^n$ to $p\mathbb{Z}^n$. It follows that for any $k$, any sufficiently long product of the same form sends $\mathbb{Z}^n$ to $p^k\mathbb{Z}^n$, so that any sufficiently long commutator in $t$ and $\gamma_j(P)/\gamma_{j+1}(P)$ is trivial.

An easy induction shows that any sufficiently long commutator in $P_\alpha$ thus lands in $\gamma_i(P)$, showing that $P_\alpha$ is nilpotent. It suffices to show that $\mu$ is a power of $p$, showing that $P_\alpha$ is a $p$–group. By assumption, we have a splitting $\mathbb{Z}/\mu\mathbb{Z} \to P_\alpha$, and the image is not normal: in fact for each $1 \leq i < \mu$, there is a nontrivial commutator of the form $[t^i, v]$ for some $v \in P$. It follows that if $\mu$ is not a power of the prime $p$, then $P_\alpha$ cannot be a product of its Sylow subgroups and is therefore not nilpotent. It follows that $P_\alpha$ is a $p$–group. $\square$

We sketch another proof of the previous lemma which relies on the theory of Lie algebras.

**Lemma 3.5.** *Let $G$ be a $p$–group, and suppose that $\psi \in \operatorname{Aut}(G)$ acts unipotently on $H_1(G, \mathbb{Z}/p\mathbb{Z})$. Then $\psi$ has $p$–power order as an automorphism of $G$.*

*Proof.* We form a semidirect product $G_\psi$, using $t$ to denote the stable letter. Suppose that $A$ is induced by $\psi$ and acts unipotently on $H_1(G, \mathbb{Z}/p\mathbb{Z})$. As before, we let $\mathcal{L}(1, p) = H_1(G, \mathbb{Z}/p\mathbb{Z})$, and we will write $\mathcal{L}(i, p)$ to be

$$\gamma_i(G)/\gamma_{i+1}(G) \otimes \mathbb{Z}/p\mathbb{Z}.$$

We will consider $\mathcal{L}(i, p)$ as a quotient of

$$\operatorname{Hom}(H_1(G, \mathbb{Z}/p\mathbb{Z}), \mathcal{L}(i, p)),$$

which we write as

$$H^1(G, \mathbb{Z}/p\mathbb{Z}) \otimes \mathcal{L}(i, p).$$

The quotient map is given by the Lie bracket (cf. [MKS], [M]). Consider the reduction of $\mathcal{L}_i(G) = \gamma_i(G)/\gamma_{i+1}(G)$ modulo $p$. Elements of $\mathcal{L}_i(G)$ are finite sums of simple tensors, which are images of simple tensors in $H^1(G, \mathbb{Z}) \otimes \mathcal{L}_i(G)$. The simple tensors which persist after reducing modulo $p$ can be written so that no $p$–multiple of a cohomology class of $G$ occurs in the tensor. It follows that the canonical map $\mathcal{L}_i(G) \to (\mathcal{L}_i(G) \pmod p)$ factors through $\mathcal{L}(i, p)$.

If $\psi$ acts unipotently on $H_1(G, \mathbb{Z}/p\mathbb{Z})$ then it also acts unipotently on the cohomology since the two actions are dual to each other. We write $A$ for the monodromy matrix acting on $H_1(G, \mathbb{Z}/p\mathbb{Z})$, $A^*$ its transpose, and $A(i)$ the associated matrix acting on $\mathcal{L}(i, p)$. Clearly 1 is the unique point in the spectrum of both $A$ and $A^*$. Suppose inductively that $A(i)$ is unipotent. Then the points of the spectrum of $A(i+1)$ are pairwise products of the points of the spectrum of $A(i)$ and $A^*$, so that $A(i+1)$ acts unipotently on $\mathcal{L}(i+1, p)$.

Filter $G$ by its lower central series, writing $\mathcal{L}(i)$ for the $i^{th}$ quotient as above. Since a sufficiently large power of $A(i) - I$ sends $\mathcal{L}(i)$ to $p\mathcal{L}(i)$, we have that forming the commutator $[t, G]$ sufficiently many times will send $G$ to a term arbitrarily deep in its lower central series. It follows that $G_\psi$ is nilpotent. But then we must have that $\psi$ has $p$–power order as an automorphism of $G$, since otherwise the nilpotence of $G_\psi$ would be contradicted as in the previous lemma. $\square$

**Lemma 3.6.** *Let $P$ be a $p$-group and $\Gamma < \operatorname{Aut}(P)$ the group of automorphisms which induce the identity on $P/\varphi(P)$. Then $\Gamma$ is a $p$-group.*

*Proof.* This is now immediate since $\Gamma$ acts trivially and hence unipotently on $\mathcal{L}_1(P)$. $\square$

We will also need the following well–known result, whose proof we give because it seems to be new.

**Lemma 3.7.** *Let $G$ be a finitely generated free group or a surface group and let $p$ be a prime. Then $G$ is residually $p$.*

*Proof.* Suppose first that $G$ is free. Identify $G$ with the fundamental group of a finite wedge of circles $X$, which we endow with the graph metric. We let $X = X_0$ and we build a tower of finite covering spaces by taking $X_{i+1}$ to be the cover of $X_i$ corresponding to $H_1(X_i, \mathbb{Z}/p\mathbb{Z})$. Note that $X_i \to X_0$ is a normal covering space of $p$–power degree.

Let $\Gamma$ be a finite graph. Let $\gamma$ be a loop in $\Gamma$, which we view as a path with the same initial and terminal vertex. We say that $\gamma$ is a simple loop in $\Gamma$ if it visits each vertex other than the initial one at most once. Note that $\gamma$ represents a nontrivial homology class modulo $p$. Note also that in any finite graph with the graph metric, any minimal length loop is always simple. It follows that any loop of length $k$ in $X_0$ does not lift to $X_k$.

For surface groups there is an additional complication, which is that short loops may be homologically trivial. Let $G = \pi_1(\Sigma)$, and we will choose a discrete, cocompact embedding $G \to PSL_2(\mathbb{R})$ so that $\Sigma$ is then a quotient of the hyperbolic plane $\mathbb{H}^2$. Each essential homotopy class of curves in $\Sigma$ is now represented by a hyperbolic geodesic. We again let $\Sigma = X_0$ and we construct the tower of $p$–power covering spaces as before.

Note that with the hyperbolic metric, any shortest closed geodesic is still simple. If $\gamma \subset X_i$ is a shortest length closed geodesic, we record the homology class of $\gamma$. If it is nontrivial then $\gamma$ is nonseparating and hence is nontrivial in $H_1(X_i, \mathbb{Z}/p\mathbb{Z})$. Otherwise, $\gamma$ is separating and each lift of $\gamma$ to $X_{i+1}$ is nonseparating, so that $\gamma$ does not lift to a closed loop in either $X_{i+1}$ or $X_{i+2}$. The length spectrum of geodesics in $X_0$ is discrete, so that as $i$ tends to infinity, the length of the shortest loop in $X_0$ which lifts to $X_i$ also tends to infinity. □

We close this section with a brief discussion of the relationship between residually $p$ groups and residually torsion–free nilpotent groups. Let $G$ be a finitely generated group, $g \in G$, and let $\mathcal{P}$ be a set of primes. We say that $g \in G$ is **$\mathcal{P}$–good** if there is an $N = N(g)$ such that $g$ survives in a $p$–group quotient of $G$ of nilpotence degree no more than $N$ for each $p \in \mathcal{P}$.

**Lemma 3.8.** *Let $G$ be a finitely generated group. The following are equivalent.*

1. *$G$ is residually torsion–free nilpotent.*
2. *There is an infinite set of prime $\mathcal{P}$ such that each nontrivial $g \in G$ is $\mathcal{P}$–good.*
3. *Each nontrivial $g \in G$ is $\mathcal{P}$–good with respect to the set of all primes.*

*Proof.* Suppose $G$ is residually torsion–free nilpotent, and let $1 \neq g \in G$. Let $N$ be a torsion–free nilpotent quotient of $G$ where $g$ survives. By Mal'cev's Theorem (see [R]), we may embed $N$ as a group $U$ of integral unipotent matrices in some finite dimensional general linear group. Let $p$ be any prime. Reducing the entries of $U$ modulo $p^n$ results in a finite $p$–group $U_{p^n}$ whose nilpotence degree is no larger than that of $N$. Choosing $n$ sufficiently large, we see that 1 implies 3. 3 implies 2 trivially. To see that 2 implies 1, let $1 \neq g \in G$ and for each $p \in \mathcal{P}$, let $P_g$ be an at most degree $k$ nilpotent

$p$–group in which $g$ survives. Write

$$M = \prod_{p \in \mathcal{P}} P_g,$$

and let $N$ be the image of $G$ in $M$. Notice that $M$ is at most $k$–step nilpotent, and $N$ is a finitely generated subgroup of $M$ in which $g$ has infinite order. The torsion elements of $N$ form a normal subgroup of $N$ (see [R]), so there is a torsion–free quotient $N'$ of $N$ in which $g$ survives. In particular, $G$ is residually torsion–free nilpotent. □

## 4. Fibered manifolds and the proof of Theorem 1.2

Given the material developed in section 3, the proof of the first part of Theorem 1.2 is not difficult. Theorem 1.2 is already not surprising since the automorphism group of a (topologically) finitely generated pro–$p$ group is already virtually a pro–$p$ group (this can in fact already be deduced from the results in section 3, cf. [DDMS]).

*Proof of Theorem 1.2, part 1.* Let $U < GL_n(\mathbb{Z}/p\mathbb{Z})$ be a unipotent subgroup. Then $U$ is a $p$–group. Let $P$ be a $p$–group quotient of $\pi_1(\Sigma)$ by a characteristic subgroup and suppose that $\alpha \in \mathrm{Aut}(P)$ acts unipotently on $H_1(\Sigma, \mathbb{Z}/p\mathbb{Z})$. Since $U$ is a $p$–group, we have that a $p$–power of $\alpha$ acts trivially on $P/\varphi(P)$. But this power of $\alpha$ has $p$–power order, so that the semidirect product $P_\alpha$ is a $p$–group. If $\alpha$ is induced by $\psi$, it follows that $P_\alpha$ is a quotient of $G_\psi$. It follows that $G_\psi$ is residually $p$, since $\pi_1(\Sigma)$ is exhausted by characteristic subgroups of $p$–power index. □

From the proofs above, we see that Theorem 1.2 is a special case of the following more general result, whose proof is identical:

**Theorem 4.1.** *Let $G$ be a finitely generated residually $p$ group and $\Gamma$ a group of automorphisms of $G$ which act unipotently on $H_1(G, \mathbb{Z}/p\mathbb{Z})$. Then $G_\Gamma$ is residually $p$, where $G_\Gamma$ is the semidirect product which fits into the following short exact sequence:*

$$1 \to G \to G_\Gamma \to \Gamma \to 1.$$

We now turn to torus bundles over the circle, whereby we can give a more complete characterization of when the fundamental group of the bundle is residually $p$. The following proposition contains the rest of the content of Theorem 1.2.

**Proposition 4.2.** *Let $A \in SL_2(\mathbb{Z})$ be hyperbolic and $M_A$ the associated torus bundle with fundamental group $G_A$. Then $G_A$ is not residually $p$ for any $p$ if and only if $A$ is conjugate over $\mathbb{Q}$ to*

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

*Furthermore, $G_A$ is residually $p$ if and only if $p \mid \det(A - I)$.*

*Proof.* We first show that $G_A$ is residually nilpotent if and only if $A$ is not in the $\mathbb{Q}$–conjugacy class of

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Suppose $A-I \in GL_2(\mathbb{Z})$. Then we can solve the equations $(A-I)v = (1,0)$ and $(A-I)w = (0,1)$ in $\mathbb{Z}^2$. If $t$ denotes the monodromy generator, it follows that $[t, \mathbb{Z}^2] = \mathbb{Z}^2$, so that the sequence of subgroups $\{\gamma_i(G)\}$ stabilizes at $i = 1$ with $\gamma_i(G) = \mathbb{Z}^2$ for all $i \geq 1$. It follows in this case that $G_A$ is not even residually nilpotent. Notice that if

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

then $A - I \in GL_2(\mathbb{Z})$. In particular $G_A$ is not residually nilpotent.

Now let $A$ be a general hyperbolic element of $SL_2(\mathbb{Z})$ with trace $n$. We will show that $G_A$ is residually $p$ if and only if $p$ divides $n-2$. If no primes divide $n-2$, it will follow that $A$ is as above. Note that over $\mathbb{R}$, $A$ is conjugate to

$$X = \begin{pmatrix} n-1 & 1 \\ n-2 & 1 \end{pmatrix}.$$

This is because every hyperbolic element of $PSL_2(\mathbb{R})$ is determined up to conjugacy by its trace. So, there exists a real matrix $Q$ such that $QA = XQ$. Finding the entries of $Q$ is tantamount to solving a system of linear equations with integer entries. Since $Q$ is only well–defined up to multiplication by a scalar matrix, we may assume that one of the entries is equal to 1. By Cramer's rule the solutions are rational, so we may assume $Q$ has rational entries. Now let $Y = X - I$. It follows that $A - I$ is conjugate to $Y$ over $\mathbb{Q}$.

One verifies that

$$Y^2 = \begin{pmatrix} (n-2)^2 + n - 2 & n-2 \\ (n-2)^2 & n-2 \end{pmatrix}.$$

In particular, each entry of $Y^k$ is divisible by $n-2$ whenever $k \geq 2$. If $n$ is at least 4 then there is a prime which divides $n-2$. Let $p$ be a prime dividing $n-2$ and let $\alpha$ be an entry of $Q$. Write $\alpha = a/b$, where $a$ and $b$ are relatively prime integers. Some power of $p$ divides $b$, and let $m$ be the maximal power of $p$ dividing the denominator of any entry of $Q$ or $Q^{-1}$. Take $k$ sufficiently large so that every entry of $Y^k$ is divisible by $p^{2m}$. Then $QY^kQ^{-1} = (A-I)^k$ and we see that every entry of $(A-I)^k$ is divisible by $p$. It follows that $A-I$ is nilpotent modulo every power of $p$. It follows that the quotient of $G_A$ given by reducing the torus homology modulo $p^k$ is nilpotent, so that $G$ is residually $p$.

Now suppose that $p$ does not divide $n-2$. Note first that $A-I$ is invertible modulo $p$. We must show that $G_A$ does not inject into its pro–$p$ completion. Consider the abelianization of $G_A$. Clearly there is an infinite cyclic summand coming from the action of the stable letter. There is also a torsion part of the abelianization, but we claim that it has trivial $p$–component, namely that it dies under tensoring with $\mathbb{Z}/p\mathbb{Z}$. This follows from the fact that $A-I$ is invertible modulo any prime and the fact that the homology of a semidirect product is identified with the group of co–invariants of the action of the stable letter. But then if $P$ is any $p$–group quotient of $G_A$, we have that the abelianization of $P$ is cyclic. By the results in Section 3, we have that $P$ is itself cyclic. □

Applying Theorem 1.2, we obtain:

**Corollary 4.3.** *Let $A \in SL_2(\mathbb{Z})$ and $p$ a prime. Then there are infinitely many $k$ such that $p \mid \det(A^k - I)$.*

**Corollary 4.4.** *Every manifold modeled on Sol geometry has a virtually residually $p$ fundamental group.*

We will see in the remainder of the paper that it seems unlikely that we can obtain a complete description of mapping classes which give rise to residually $p$ fibered manifolds which are not torus bundles.

Here we make two remarks concerning generalizations of the previous proposition. Firstly, there is a similar result about higher dimensional torus bundles which can be found in a preprint of M. Aschenbrenner and S. Friedl, namely [AF2]. They consider a torus bundle built out of a map $\phi$ whose action on the homology of the torus has characteristic polynomial $q_\phi$. They show that the resulting semidirect product is residually $p$ if and only if each irreducible factor $q$ of $q_\phi$ satisfies $q(1) \equiv 0 \pmod{p}$. It is easy to see that Aschenbrenner and Friedl's result agrees with the discussion above, and it is not difficult to see how one might recover their result by similar methods.

The second remark is that much of this discussion can be easily generalized to mere endomorphisms of finitely generated abelian groups, not just automorphisms.

## 5. Finite-volume hyperbolic orbifolds

The main external tool for proving Theorem 1.3 is the following result recently announced by Wise in [W1] (cf. [W2], [WK]):

**Theorem 5.1.** *Let $G$ be a word-hyperbolic group with a quasiconvex hierarchy. Then $G$ has a finite index subgroup $G'$ which embeds in a graph group $R$.*

The precise meanings of all the terms in Theorem 5.1 are not important for our purposes. However, two important (though non–obvious) corollaries of a relative version of Theorem 5.1 (also announced in [W1]) are the following:

**Corollary 5.2.** *Let $M$ be a finite volume cusped hyperbolic manifold, and assume $M$ contains a geometrically finite incompressible surface $S$. Then $G = \pi_1(M)$ has a finite index subgroup $G'$ which embeds in a right-angled Artin group.*

A Haken hierarchy for $M$ implies the existence of a hierarchy for $\pi_1(M)$. By the work of Thurston, this is a quasiconvex hierarchy if and only if $S$ is geometrically finite (see [T], cf. [S]).

**Corollary 5.3.** *Every Haken hyperbolic 3–manifold is virtually fibered.*

*Proof.* By the work of Bonahon in [B], an incompressible surface $S$ is either a virtual fiber, or $S$ is geometrically finite. In the first case, $M$ virtually fibers. In the second case, there is a finite cover $\widehat{M}$ of $M$ such that $\pi_1(\widehat{M}) < R$ for a graph group $R$. Each graph group is residually finite rationally solvable. By [A2], $M$ virtually fibers. □

Let $\Gamma$ be as in the statement of Theorem 1.3. Any such $\Gamma$ is the fundamental group of a hyperbolic orbifold. After passing to a finite cover, we may assume that $\Gamma$ is the fundamental group of a hyperbolic 3–manifold $M$ of finite volume. By a result due to Thurston, the representation $\pi_1(M) \to PSL_2(\mathbb{C})$ lifts to $SL_2(\mathbb{C})$ (cf. [CS]).

Let $\mathcal{R} = \mathcal{R}(\Gamma)$ denote the $SL_2(\mathbb{C})$ representation variety of $\Gamma$. By general theory (see [R] for instance, though this theory goes back to the work of Mal'cev), $\mathcal{R}$ contains a point over $\overline{\mathbb{Q}}$ and in fact a faithful representation $\Gamma \to SL_2(\overline{\mathbb{Q}})$. Since $\Gamma$ is finitely generated, there is a finite extension $K/\mathbb{Q}$ such that the image of $\Gamma$ lands in $SL_2(K)$. We let $\mathcal{O}$ denote the ring of integers in $K$. In any matrix in the image, there are at most four denominators, and so any finite generating set for $\Gamma$ has only finitely many denominators occurring among nonzero entries in its image. Fix a finite generating set for $\Gamma$ and consider the denominators which occur. These will be contained in finitely many prime ideals in $\mathcal{O}$. Each prime ideal of $\mathcal{O}$ lies over a unique prime ideal $p\mathbb{Z}$. For the set of denominators which occur in the image a generating set for $\Gamma$, let $\mathcal{B} \subset \mathbb{Z}$ be the finite set of primes over which the associated prime ideals in $\mathcal{O}$ lie. We call $\mathcal{B}$ the set of **bad primes**.

**Lemma 5.4.** *Let $p$ be any prime. When $\mathcal{B}$ is empty, $\Gamma$ is virtually residually $p$.*

*Proof.* This is entirely analogous to the fact that $SL_2(\mathbb{Z})$ is virtually residually $p$ and is done using the first congruence subgroup. Let $P$ be a prime ideal which lies over $p\mathbb{Z}$. Let $\Gamma_1$ denote the kernel of the natural map

$$SL_2(\mathcal{O}) \to SL_2(\mathcal{O}/P).$$

We have a natural action of $\Gamma_1$ on $\mathcal{O}^2$, and we can construct the semidirect product

$$1 \to \mathcal{O}^2 \to G \to \Gamma_1 \to 1.$$

We can also construct truncated semidirect products of the form

$$1 \to (\mathcal{O}/P^n)^2 \to G_n \to \Gamma_1 \to 1.$$

By considering the successive quotients $(P^i/P^{i+1})^2$, we see that the conjugation action of $\Gamma_1$ on $(\mathcal{O}/P^n)^2$ is unipotent. Let $K_n < \Gamma_1$ denote the kernel of this action. We have that $\mathcal{O}/P^n$ is always a $p$–group. It follows that the semidirect product

$$1 \to (\mathcal{O}/P^n)^2 \to \overline{G}_n \to \Gamma_1/K_n \to 1$$

is a $p$–group, so that

$$\bigcap_n K_n = \{1\}$$

and $\Gamma_1/K_n$ is a $p$–group for all $n$ (a similar argument is fleshed out in Section 3, cf. [BL]). $\square$

Alternatively, the argument could have proceeded as follows: $(\mathcal{O}/P^n)^2$ is a $p$–group, and $(P/P^n)^2$ is its Frattini subgroup. On the other hand, we have seen that if $Q$ is a $p$–group and $\phi(Q)$ is its Frattini subgroup, then the group of automorphisms of $Q$ which induce the identity on $Q/\phi(Q)$ form a $p$–group, whence the conclusion.

We now consider the case where $\mathcal{B} \neq \emptyset$. Fix $P \in \mathcal{B}$, and let $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_P$ be the completion of $\mathcal{O}$ at $P$, namely

$$\widehat{\mathcal{O}} = \varprojlim \mathcal{O}/P^n.$$

We let $\widehat{K}$ be the fraction field of $\widehat{\mathcal{O}}$. We have a canonical map $\mathcal{O} \to \widehat{\mathcal{O}}$ which is injective since $\mathcal{O}$ is a Dedekind domain, and thus we have an injective map $K \to \widehat{K}$, and a faithful representation $\Gamma \to SL_2(\widehat{K})$ induced by the inclusion $SL_2(K) \to SL_2(\widehat{K})$. By construction, the field $\widehat{K}$ comes equipped with a discrete valuation $\nu$. Explicitly, it takes an equivalence class of fractions $\gamma = \alpha/\beta$, determines an $i$ and $j$ such that $\alpha \in P_i \setminus P^{i+1}$ and $\beta \in P^j \setminus P^{j+1}$ and sets $\nu(\gamma) = i - j$. By abuse of notation, we write $P$ as the maximal ideal generated by the image of $P$ in $\widehat{\mathcal{O}}$. The valuation $\nu$ thus defined is a discrete valuation, so that $\widehat{\mathcal{O}}$ is a DVR.

Let $V$ be a two–dimensional vector space over $\widehat{K}$. Recall that at $\widehat{\mathcal{O}}$–lattice in $V$ is a rank two $\widehat{\mathcal{O}}$–module which spans $V$ as a $\widehat{K}$–vector space. Let $L$ be a $\widehat{\mathcal{O}}$–lattice and $L'$ a sublattice. Then $L/L'$ is isomorphic to $\widehat{\mathcal{O}}/P^a \oplus \widehat{\mathcal{O}}/P^b$ for some choice of nonnegative integers $a$ and $B$. There is a natural action of $\widehat{K}$ on the set of $\widehat{\mathcal{O}}$–lattices in $V$. Note that if $L$ and $L'$ are arbitrary lattices, we can replace $L'$ by an equivalent lattice $kL'$ such that $kL' \subset L$ by choosing an appropriate $k \in \widehat{K}$. We declare the $\widehat{K}$–orbits to be equivalence classes. There is a natural graph whose vertices are equivalence classes of lattices, and whose edges span pairs of equivalence classes for which there exist representatives satisfying $L/L' \cong \widehat{\mathcal{O}}/P$. It is shown in [Se] that this graph is a tree, called the lattice tree of $\widehat{\mathcal{O}}$.

We have that $SL_2(\widehat{K})$ acts on this tree in the obvious way. The stabilizers of vertices are precisely the $GL_2(\widehat{K})$–conjugates of $SL_2(\widehat{\mathcal{O}})$ in $SL_2(\widehat{K})$. The following lemma follows easily from this discussion.

**Lemma 5.5.** *If $\Gamma$ is as above, then either $\Gamma$ is virtually residually $p$ or $\Gamma$ acts on the lattice tree of $\widehat{\mathcal{O}}$ without a global fixed point.*

General Bass–Serre theory (cf. [Se]) therefore implies that when $\Gamma$ acts nontrivially on the lattice tree, then $\Gamma$ splits as a nontrivial amalgamated product. Furthermore, the amalgamating group can, up to conjugacy be taken to be the image of $\Gamma$ in $SL_2(\widehat{\mathcal{O}})$. Since $\Gamma$ is the fundamental group of a hyperbolic manifold, the amalgamating group is nontrivial. Indeed, $\mathbb{H}^3/\Gamma$ is irreducible and hence $\Gamma$ cannot split as a nontrivial free product.

The final ingredient we need is the following, which is due to Epstein, Stallings and Waldhausen, and a proof can be found in [CS].

**Lemma 5.6.** *Let $M$ be a compact, orientable 3–manifold. For any nontrivial splitting of $\pi_1(M)$ there exists a nonempty system $S$ of incompressible non-peripheral surfaces such that the image of the inclusion on fundamental groups is contained in an edge group. Furthermore, the image of the fundamental groups of the components of $M \setminus S$ are contained in a vertex group.*

This lemma applies to our situation, since by [A1] and [CG], $\mathbb{H}^3/\Gamma$ is homeomorphic to the interior of a compact 3-manifold. From here, Theorem 1.3 is obvious:

*Proof of Theorem 1.3.* If $\Gamma$ cannot be coaxed into admitting a faithful representation into $SL_2(\mathcal{O})$, we have that $\mathcal{B} \neq \emptyset$. For each $p \in \mathcal{B}$, choose $P \subset \mathcal{O}$ lying over $p$. We constructed a faithful representation of $\Gamma$ into $SL_2(\widehat{K})$

whose image does not lie in $SL_2(\widehat{\mathcal{O}}_P)$. But then we obtain a nontrivial splitting of $\Gamma$, and conclude that $\mathbb{H}^3/\Gamma$ is Haken. By assumption, any virtually Haken hyperbolic orbifold is virtually fibered. We have already shown that any 3–manifold which virtually fibers over the circle has a virtually residually $p$ fundamental group. $\square$

We briefly remark that whereas general theory says that a finitely generated subgroup of a linear group over a characteristic zero field is always residually $p$ for all but finitely many primes, it may not be residually $p$ for all primes. An example, furnished in [W], is the metabelian group generated by

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

contains a subgroup isomorphic to the dyadic rationals and hence cannot be residually $p$ at 2.

To give more content to the discussion of 3–manifold groups which are residually $p$, it would be good to see that there are hyperbolic 3–manifolds which are not already residually $p$ for some prime. Perhaps the easiest examples of such manifolds come from hyperbolic knot complements:

**Proposition 5.7.** *Let $M = S^3 \setminus K$ be a nontrivial knot complement. Then $\pi_1(M)$ is not residually nilpotent. In particular, $\pi_1(M)$ is not residually $p$ for any prime.*

*Proof.* It is well–known that $H_1(M, \mathbb{Z}) \cong \mathbb{Z}$. It follows that the abelianization of $\pi_1(M)$ is cyclic, so that $\pi_1(M)$ admits no nonabelian nilpotent quotients. $\square$

## 6. Applications to the mapping class group and the proof of Theorem 1.7

The original motivation for the work in this paper was to study homological representation theory of the mapping class group, as initiated in [K]. In that paper, the author constructs an infinite dimensional representation $H(\Sigma)$ of the marked mapping class group $\mathrm{Mod}^1(\Sigma)$, which is constructed by identifying $\mathrm{Mod}^1(\Sigma)$ with a subgroup of $\mathrm{Aut}(\pi_1(\Sigma))$, taking any family of characteristic covers of the surface $\Sigma$, and letting the mapping classes act on the homology of these covers. The content of [K] is that if homology is taken with complex coefficients and the family of covers exhausts $\pi_1(\Sigma)$, then the representation is faithful and detects the Nielsen–Thurston classification of each mapping class.

The representation $H(\Sigma)$ is rather difficult to understand, and the hope was that the fundamental group of $M_\psi$ would lead to some insight into the action of $\psi$ on $H(\Sigma)$. One reason for this belief is Theorem 1.7, which says that if $G_\psi$ is residually $p$ and $\phi$ acts trivially on the modulo $p$ homology of the fiber, then $G_{\psi\circ\phi}$ is also residually $p$. In other words, whether or not $G_\psi$ is residually $p$ depends only on the image of $\psi$ in the modulo $p$ homology representation $\rho : \mathrm{Mod}(\Sigma) \to GL_n(\mathbb{Z}/p\mathbb{Z})$. One would expect that if a good criterion for $G_\psi$ to be residually $p$ could be devlopped and could be read off from $\rho(\psi)$, one might be able to restrict the possible actions of $\psi$ on $H(\Sigma)$.

We will now provide a proof of Theorem 1.7 and then give some examples which illustrate that a good criterion might not exist.

*Proof of Theorem 1.7.* This is a manifestation of the fact that for any $p$–group $P$, the group of automorphisms which act trivially on $P/\varphi(P)$ is a $p$–group. Let $P$ be a $p$–group quotient of $G_\psi$, and let $P'$ be the image of $\pi_1(\Sigma)$ inside of $P$. Note that $\psi$ and $\psi\circ\phi$ act in the same way on $H_1(\Sigma,\mathbb{Z}/p\mathbb{Z})$, so that the action of these two automorphisms coincides on $P'/\varphi(P')$. Let $K$ be the kernel of the map $\pi_1(\Sigma)\to P'/\varphi(P')$. Since $P$ is nilpotent, it follows that there is a $k$ such that $\gamma_k(G_{\psi\circ\phi}) < K$. The claim that every element of $G_\psi$ which survives in $P$ survives in a $p$–power quotient of $G_{\psi\circ\phi}$ follows by induction on the order of $P'$. □

Note that if $\phi$ is as in the hypotheses of Theorem 1.7, it is immediate that $G_\psi$ is residually $p$ if and only if $G_{\psi\circ\phi}$ is residually $p$.

## 7. Central extensions of residually $p$ groups

The one geometry we have not considered to this point is $\widetilde{PSL_2}(\mathbb{R})$ geometry. If $\Gamma$ is a finitely generated discrete subgroup of isometries of this geometry then $\Gamma$ fits into a short exact sequence of groups as follows (cf. [T2]):

$$1\to\mathbb{Z}\to\Gamma\to H\to 1.$$

Furthermore, the copy of $\mathbb{Z}$ in the sequence is central. The group $H$ is a discrete subgroup of isometries of $\mathbb{H}^2$, so that virtually $\Gamma$ is a $\mathbb{Z}$–central extension of the fundamental group of a hyperbolic surface and hence (virtually) the fundamental group of an orientable circle bundle over a surface. Recall that central extensions of a group $G$ by a group $A$ have exactly one obstruction to triviality, namely the Euler class, which is an element of $H^2(G,A)$ (see [Br]). It follows immediately that:

**Proposition 7.1.** *Let $H$ be as above, and suppose that $H$ is virtually $\pi_1(\Sigma)$. If $\Sigma$ is not closed then $\Gamma$ virtually splits as a product of $\mathbb{Z}$ and a free group. In particular, $\Gamma$ is the fundamental group of a virtually fibered 3-manifold and is hence virtually residually $p$.*

If $\Sigma$ is closed, it is not immediate that $\Gamma$ is residually $p$. In fact, we have a more general question:

**Question 7.2.** *Let $P$ be a finite abelian $p$–group which fits into a short exact sequence*

$$1\to P\to G\to Q\to 1,$$

*where $Q$ is residually $p$. Suppose that the conjugation action of $G$ on $P/\varphi(P)$ is unipotent. Under what conditions is $G$ residually $p$?*

Since $G$ acts unipotently on $P/\varphi(P)$ and since $P$ is finite, we obtain a $p$–power index normal subgroup $G'$ of $G$ such that

$$1\to P\cap G'\to G'\to Q'\to 1$$

is a central extension. We can also consider the case where $P$ is replaced by a torsion–free abelian group and the extension is assumed to be central from the start.

Notationally, we replace $G'$ by $G$. Suppose that $\gamma \in P$ and we want a finite $p$–group quotient $P'$ of $G$ such that $\gamma$ survives in the quotient. If $P$ is finite, we may assume that the central extension

$$1 \to P \to G \to Q \to 1$$

descends to another central extension

$$1 \to P \to P' \to Q' \to 1.$$

Here, $Q'$ is a $p$–group quotient of $Q$. We have that the extension $P'$ is classified by an element of $H^2(Q', P)$.

We have a natural pullback map $H^2(Q', P) \to H^2(Q, P)$, and we can hope for $G$ to be residually $p$ only if the classifying cocycle for

$$1 \to P \to G \to Q \to 1$$

is contained in the image of $H^2(Q', P)$. Let us be a bit more explicit about this fact. The standard discussion we follow here can be found in [Br]. Let $G$ be an arbitrary group and let $A$ be an abelian group equipped with the structure of a trivial $G$-module. It is classical that there is a bijection between the sets $E(G, A)$ of central extensions of $G$ by $A$ and $H^2(G, A)$. This bijection associates to each cohomology class $c \in H^2(G, A)$ a function $f = f_c : G \times G \to A$ satisfying the cocycle condition, called the **factor function**. If we are given an extension

$$1 \to A \to E \to G \to 1,$$

we choose a set-theoretic section $s : G \to E$. We normalize $s$ so that $s(1) = 1$. If $i : A \to E$ is the inclusion map, we define $f$ by $i(f(g, h)) = s(g)s(h)s(gh)^{-1}$. The group law on $A \times G$ can be recovered by $(a, g)(b, h) = (a + b + f(g, h), gh)$.

Let $F$ be a quotient of $E$ with quotient map $\phi$, and suppose that the induced map $A \to F$ is injective. Since $A$ is central, $\phi$ descends to a quotient map $G \to F/A$ which we also call $\phi$. The question in which we are really interested here is when a given quotient map $\phi : G \to F/A$ extends to a quotient map $\phi : E \to F$. Then, we may write

$$1 \to A \to F \to F/A \to 1,$$

and $F/A$ is a quotient of $G$. A group law on $A \times F/A$ must be given by $(a, \phi(g))(b, \phi(h)) = (a + b + k(\phi(g), \phi(h)), \phi(gh))$ for some factor function $k$. If $F$ is to be a quotient of $E$, we must have $(a + b + k(\phi(g), \phi(h)), \phi(gh)) = (a + b + f(g, h), \phi(gh))$. We have that $f$ determines an element of $H^2(G, A)$ and $k \circ \phi$ determines another element of $H^2(G, A)$ which is a pullback of an element of $H^2(F/A, A)$. All the equation $(a + b + k(\phi(g), \phi(h)), \phi(gh)) = (a + b + f(g, h), \phi(gh))$ says is that $f = \phi^* k$.

The main result of all this discussion will be:

**Proposition 7.3.** *Let $S^1 \to M \to \Sigma$ be a nontrivial orientable circle bundle with trivial monodromy over $\Sigma$ and let $p$ be a prime. Then $\Gamma = \pi_1(M)$ is virtually residually $p$.*

In Section 8 we shall develop technically simpler characteristic zero machinery to prove this proposition as a corollary to Theorem 1.5, which will show that $\Gamma$ is residually torsion–free nilpotent.

## 8. RESIDUALLY TORSION-FREE NILPOTENT 3-MANIFOLD GROUPS

In [Wi], Wilton classifies the fundamental groups of 3–manifolds which are residually free, in other words which 3–manifold groups are residually free groups. He shows that residually free groups among 3–manifold groups are rare:

**Proposition 8.1** ([Wi]). *If $M$ is a prime, compact 3-manifold with incompressible torus boundary, then $\pi_1(M)$ is residually free and nontrivial if and only if $M$ is one of the following:*

1. *A trivial circle bundle over an orientable surface.*
2. *A circle bundle with trivial monodromy over a non-orientable surface of Euler characteristic less than $-1$.*
3. *The nontrivial circle bundle with trivial monodromy over the projective plane.*

He also asks:

**Question 8.2.** *Which (closed) 3–manifolds have fundamental groups which are residually torsion–free nilpotent?*

In this section we adapt the previous discussion to explore this question. In the introduction, we claimed that if $\psi \in \mathrm{Mod}(\Sigma)$ and acts unipotently on $H_1(\Sigma, \mathbb{Z})$, then $\pi_1(M_\psi)$ is residually torsion-free nilpotent. We now give a proof of this fact:

*Proof of Theorem 1.4.* In light of the theory we have developed so far in this paper, the claim is almost obvious. Let $H$ denote $\pi_1(\Sigma)$ and lift $\psi$ arbitrarily to $\mathrm{Aut}(H)$. The argument in Lemma 3.5 implies that $\psi$ acts unipotently on $\gamma_i(H)/\gamma_{i+1}(H)$ for all $i$.

Since the action on $H_1(\Sigma)$ is unipotent, it follows that the semidirect product

$$1 \to H_1(\Sigma) \to N_1 \to \mathbb{Z} \to 1$$

is nilpotent, where the conjugation action of $\mathbb{Z}$ is given by $\psi$. The kernel of the map $\pi_1(M_\psi) \to N_1$ is precisely $\gamma_1(H)$.

By induction, we assume that the semidirect product

$$1 \to H/\gamma_i(H) \to N_i \to \mathbb{Z} \to 1$$

is a nilpotent group, where the conjugation action of $\mathbb{Z}$ is again given by $\psi$. The action of $\psi$ on $\gamma_i(H)/\gamma_{i+1}(H)$ is unipotent, so that the semidirect product

$$1 \to \gamma_i(H)/\gamma_{i+1}(H) \to Z_i \to \mathbb{Z} \to 1$$

is also nilpotent and hence has abelianization of rank at least two. Note also that conjugation within $H$ acts trivially on $\gamma_i(H)/\gamma_{i+1}(H)$. This implies that the semidirect product

$$1 \to H/\gamma_{i+1}(H) \to N_{i+1} \to \mathbb{Z} \to 1$$

is nilpotent. Indeed, the inductive hypothesis shows that there is a $k$ such that

$$\gamma_k(N_{i+1}) < \gamma_i(H)/\gamma_{i+1}(H).$$

But then since the conjugation action of $H$ is trivial on $\gamma_i(H)/\gamma_{i+1}(H)$ and since $Z_i$ is nilpotent, we have that there is a $k' \geq k$ such that $\gamma_{k'}(N_{i+1})$ is trivial.

To complete the proof, we need only show that each $N_i$ is torsion free. But this is obvious: we write $1 \neq n \in N_i$ as $t^k h$, where $h \in H/\gamma_{i+1}(H)$. We may obviously suppose that $k \neq 0$, since $\gamma_i(H)/\gamma_{i+1}(H)$ is torsion-free for all $i$ and since

$$\bigcap_i \gamma_i(H) = \{1\},$$

(cf. [MKS]). But then we just use the usual homomorphism to $\mathbb{Z} = \langle t \rangle$. We see that $\pi_1(M_\psi)$ is residually torsion–free nilpotent. □

It is well–known that in each coset of each non–central normal subgroup of the mapping class group, there are pseudo-Anosov homeomorphisms whose suspensions are hyperbolic by a well–known result of Thurston.

**Corollary 8.3.** *There exist many finite volume hyperbolic* 3*-manifolds with residually torsion–free nilpotent fundamental groups.*

We remark briefly that the conclusion of Theorem 1.4 for mapping classes in the Torelli group could have been deduced from classical work of Johnson on the residual torsion–free nilpotence of the Torelli group. Indeed, the fundamental group of a fibered 3–manifold with Torelli monodromy sits naturally as a subgroup of the Torelli group with one marked point.

By [Wi], the fundamental group of any finite volume hyperbolic 3–manifold is not a residually free group. Since there are finite volume hyperbolic 3–manifolds whose fundamental groups are residually $p$ at no primes, it follows that there are finite volume hyperbolic 3-manifolds whose fundamental groups are not residually torsion–free nilpotent. On the other hand, if one could show that all hyperbolic 3–manifolds have virtually residually torsion–free nilpotent fundamental groups then one could resolve the "virtual $b_1 > 1$ conjecture".

It follows easily from the argument in the proof of Theorem 1.4 that torus bundles with unipotent monodromy have torsion–free nilpotent fundamental groups. That the fundamental group of such a torus bundle is torsion–free also follows from Theorem 1.2, which shows that any such torus bundle is residually $p$ for every prime. A finitely generated nilpotent group is residually $p$ if and only if its torsion subgroup is a $p$–group.

**Proposition 8.4.** *A hyperbolic torus bundle over the circle is not residually torsion-free nilpotent.*

*Proof.* We have argued through Theorem 1.2 that hyperbolic torus bundles are never residually $p$ for all primes. Alternatively, let $G$ be the fundamental group of a hyperbolic torus bundle. Then $G^{ab}$ has rank one. Furthermore, the monodromy acts irreducibly on $H_1(S^1 \times S^1, \mathbb{Q})$, and it is easy to check that $[G, \pi_1(S^1 \times S^1)]$ has finite index in $\pi_1(S^1 \times S^1)$. In particular, there can be no further torsion–free nilpotent quotient of $G$. □

Suppose that $M_\psi$ is a fibered 3-manifold with fundamental group $G$ and let $N$ be a torsion-free nilpotent quotient of $G$. Let $\phi : G \to N$ be the quotient map. If $G' < G$ is a finite index subgroup, we may restrict $\phi$ to $G'$ to obtain a finite index subgroup $N' < N$ which is again nilpotent and torsion-free. It follows that being residually torsion-free nilpotent is invariant under taking finite index subgroups. The author has given examples of mapping classes in the Torelli group (pure braids, the Torelli group of a multiply punctured disk) which lift to a finite cover of the base surface and act non–unipotently in [K]. The suspensions of such mapping classes have residually torsion–free nilpotent fundamental groups, so that there can be no naïve converse to Theorem 1.4. Another complicating factor is that when $\psi$ is a Torelli mapping class, then $b_1(M_\psi) = b_1(\Sigma) + 1$, so that $M_\psi$ will admit infinitely many inequivalent fibrations (see [T3]). Furthermore, if such a fibration has fiber $\Sigma'$ with $b_1(\Sigma') > b_1(\Sigma)$, then the monodromy of that fibration cannot be in the Torelli group, and it seems unlikely that one would have so much control as to have all the fibrations to have unipotent homological monodromy.

Wilton's result shows that orientable geometric 3–manifolds whose fundamental groups are nontrivial residually free groups must admit geometric structures modeled on $S^2 \times \mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{H}^2 \times \mathbb{R}$. As for nontrivial residually torsion–free nilpotent 3–manifold groups, we have additionally exhibited examples with Nil and $\mathbb{H}^3$ geometry. $S^3$ geometry is ruled out by nontriviality, and we have ruled out Sol geometry above. We finally consider $\widetilde{PSL_2}(\mathbb{R})$ geometry and give the promised proof of Theorem 1.5:

*Proof of Theorem 1.5.* We have that $G$ fits into a non–split central extension

$$1 \to \mathbb{Z} \to G \to \pi_1(\Sigma) \to 1.$$

Killing the central copy of $\mathbb{Z}$, we see that for each $g \in G$ which projects nontrivially to $\pi_1(\Sigma)$, we have $1 \neq g \in N$ for some $N \cong \pi_1(\Sigma)/\gamma_i(\pi_1(\Sigma))$. These are all torsion–free, as is well–known from the work of Magnus.

Let $N$ be a torsion–free nilpotent quotient of $\pi_1(\Sigma)$. There is a natural map $H^2(N, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z})$ given by pullback. If $e$ is in the image of the pullback, then there is a quotient $Q$ of $G$ which fits into a central extension

$$1 \to \mathbb{Z} \to Q \to N \to 1.$$

Let $g$ be the genus of $\Sigma$, and fix a complex structure on $\Sigma$ together with a basepoint. There is a canonical isomorphism $H_1(\Sigma, \mathbb{Z}) \cong H_1(\mathbb{Z}^{2g}, \mathbb{Z})$ given by the period map. By duality we obtain an isomorphism $H^1(\mathbb{Z}^{2g}, \mathbb{Z}) \to H^1(\Sigma, \mathbb{Z})$.

There is an induced map $H^2(\mathbb{Z}^{2g}, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z})$. We have that $H_2(\mathbb{Z}^{2g}, \mathbb{Z})$ is naturally identified with $\Lambda^2 H_1(\Sigma, \mathbb{Z})$ (cf. [GH], for instance). The map $H^2(\mathbb{Z}^{2g}, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z})$ is given by taking $\alpha, \beta \in H_1(\Sigma, \mathbb{Z})$, taking their Poincaré duals $\alpha^*$ and $\beta^*$, taking $\alpha \wedge \beta \in H_2(\mathbb{Z}^{2g}, \mathbb{Z})$ and sending it to $\alpha^* \cup \beta^*$, which gives a linear function $H_2(\mathbb{Z}^{2g}, \mathbb{Z}) \to \mathbb{Z}$ (hence a second cohomology class) and produces an element of $H^2(\Sigma, \mathbb{Z})$. In particular, the map $H^2(\mathbb{Z}^{2g}, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z})$ is surjective.

It follows that if $z$ is a nontrivial element of the central copy of $\mathbb{Z}$ then $z$ is nontrivial in a torsion–free nilpotent quotient $Q$ of $G$ which can be described via

$$1 \to \mathbb{Z} \to Q \to \mathbb{Z}^{2g} \to 1.$$

In particular, each element of $G$ survives in a torsion–free nilpotent quotient of $G$. $\square$

**Corollary 8.5.** *If $G$ is as in Theorem 1.5 and $p$ is a prime, then $G$ is residually $p$.*

**Corollary 8.6** (Compare [dlH], IV.B.48)**.** *If $G$ is as in Theorem 1.5 then $G$ is linear.*

*Proof.* We have a faithful homomorphism from $G$ to a product of a surface group and a finitely generated torsion-free nilpotent group, both of which are linear. $\square$

Though the proof of Theorem 1.5 might seem a little ad hoc, there are good reasons why the quotient of $\pi_1(\Sigma)$ we consider is the abelianization. Indeed, let $C = [\pi_1(\Sigma), \pi_1(\Sigma)]$ and let $N$ be a torsion–free nilpotent quotient of $\pi_1(\Sigma)$. Suppose we have a torsion–free nilpotent quotient $N'$ of $G < PSL_2(\mathbb{R})$ which fits into a central of the form

$$1 \to \mathbb{Z} \to N' \to N \to 1,$$

where $G$ is a central extension of $\pi_1(\Sigma)$. Since $C$ is a subgroup of $\pi_1(\Sigma)$, there is also a central extension of the form

$$1 \to \mathbb{Z} \to G' \to C \to 1.$$

Since this central extension is classified by a second cohomology class of $C$, the extension splits. Indeed, $C$ is free and hence has cohomological dimension zero (alternatively $C$ is a free and hence projective object in the category of groups so that any such extension must split). It is therefore not surprising that the abelianization map on $\pi_1(\Sigma)$ would be sufficient for witnessing the residual torsion–free nilpotence of $G$.

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