DESCRIBING ALL BI–ORDERINGS ON THOMPSON'S GROUP F

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In this article, the authors explicitly describe all bi–invariant orderings on Thompson's group F. A group G is called (left) orderable if there exists a total ordering $\leq$ on G which is compatible with the group law, in the sense that $g \leq h$ and $k \in G$ implies $k \cdot g \leq k \cdot h$. A group is bi–orderable if it admits an ordering which is both left and right invariant. Equivalently, G is bi–orderable if it is left orderable by $\leq$ and $\leq$ is conjugation–invariant.

A fundamental and usually difficult question in group theory is to take a finitely generated group G and describe the set of all (bi–) orderings on G. Precisely, one identifies the ordering $\leq$ with the cone of positive elements under $\leq$, namely those greater than the identity. Thus an ordering is identified with a subset of G, and all orderings on G are identified with a subset of the power set $\mathcal{P}(G)$. The power set has a natural topology coming from the discrete topology on $\{0,1\}$ and the product topology on

$$\{0,1\}^G.$$

In this topology, the set of (bi–) orderings $O(G)$ forms a closed subset. Describing the bi–orderings on G means describing all possible conjugation-invariant positive cones in G. A related question is to describe the topology of $O(G)$. Generally, $O(G)$ is a compact set with some isolated points, some accumulation points and some Cantor sets.

There are cases when $O(G)$ is finite, as was completely described by Tararin. Noncyclic free abelian groups have a Cantor set worth of bi–orderings. For residually torsion-free nilpotent groups (for instance free groups), it is easy to produce large Cantor sets of bi–orderings. Determining whether free groups have isolated bi–orderings remains an open problem.

The present article classifies the bi–orderings on Thompson's group F. Recall that this is the group of orientation-preserving piecewise-linear homeomorphisms of the interval which induce a bijection of the dyadic rational numbers and whose derivative on each linearity interval is an integral power of 2. A theorem of Dlab shows that there are exactly four bi–orderings on the commutator subgroup $[F,F]$.

The main result of the paper is that there are exactly eight exotic isolated bi–orderings which in some sense arise from bi–orderings on $[F,F]$, together with four Cantor sets which arise from arbitrary orderings on $\mathbb{Z}^2 = F^{ab}$.

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