

# RELATIVE WEIGHT FILTRATIONS ON COMPLETIONS OF MAPPING CLASS GROUPS

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The goal of the paper is to introduce topologists to relative weight filtrations on completions of groups with a view towards applications to mapping class group theory and 3-manifold theory, building on work of Morita. An essentially complete list of references to prior work is contained in the bibliography. The structure of this paper is two-fold. First, the author introduces the audience to weight filtrations on completions of groups. Then, the author uses the theory he develops to prove results of interest to topologists.

One of the important applications of the theory discussed in the paper is in the study of the handlebody subgroup. A handlebody is determined by a maximal collection of disjoint, simple closed curves on a surface. These curves determine a so-called relative weight filtration on the Lie algebra of the completed mapping class group. The author shows that the associated relative weight filtrations of two such maximal collections are the same if and only if the two handlebodies are diffeomorphic by a diffeomorphism which restricts to the identity on the boundary. Another result characterizes the handlebody subgroup and identifies the mapping classes which extend to some handlebody.

The author mentions some potential applicability of the theory to finite-type invariants of 3-manifolds but does not expand on these ideas significantly in the present work. The reader is referred to some manuscripts which illustrate the difficulties coming from non-reductive geometric invariant theory arising in the construction of these invariants.

In the first part of the paper, the author sacrifices a self-contained discussion for one which does not obfuscate the main points with Hodge and Galois-theoretic details. The main construction is the relative completion of a discrete group. Precisely, this is a proalgebraic group which is constructed from a discrete group  $\Gamma$ , a field  $F$  of characteristic zero, a reductive algebraic group  $R$  over  $F$  (the primary examples in the paper are  $\mathrm{Sp}_n(F)$ ,  $\mathrm{GL}_n(F)$  and  $\mathrm{SL}_n(F)$ ), and a Zariski dense homomorphism  $\rho : \Gamma \rightarrow R$ .

The completion of  $\Gamma$  relative to  $\rho$  is a proalgebraic group  $\mathcal{G}$  defined over  $F$  which fits into an extension

$$1 \rightarrow \mathcal{U} \rightarrow \mathcal{G} \rightarrow R \rightarrow 1,$$

where  $\mathcal{U}$  is pronipotent. The homomorphism  $\rho : \Gamma \rightarrow R$  extends to  $\widehat{\rho} : \Gamma \rightarrow \mathcal{G}$ . The completion of  $\Gamma$  satisfies a natural universal property. The author states various properties of relative completions, and refers the reader to proofs in literature. The properties of relative completions allow him to explicitly compute certain relative completions, and he does so for irreducible lattice in semisimple real Lie groups of higher rank and multiples of the universal central extension of  $\mathrm{Sp}_g(\mathbb{Z})$ . The completion of the

mapping class group is constructed, and it is roughly an extension of the automorphism group of the real homology of the surface by the pronipotent completion of the Torelli group.

The author describes the basics of weight filtrations on vector spaces associated to nilpotent endomorphisms, and then constructs natural weight filtrations on the Lie algebra of the completed mapping class group associated to a system of simple closed curves on the surface. Throughout, he stresses the exactness and naturality properties of relative weight filtrations, together with their computability.

The second part of the paper consists of applying the machinery of relative completions of groups to mapping class groups. This is where the author obtains the topological results.

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