THE COARSE CLASSIFICATION OF COUNTABLE ABELIAN GROUPS

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One of the most fundamental problems in geometric group theory is to take a class of groups and break it up into quasi–isometry classes. Recall that two metric spaces X and Y are quasi–isometric if there is a map $f : X \to Y$ and constants $\lambda \geq 1$ and $C, D \geq 0$ such that for every pair $a, b \in X$ and every $y \in Y$,

$$\frac{1}{\lambda} d_X(a,b) - C \leq d_Y(f(a), f(b)) \leq \lambda d_X(a,b) + C,$$

and there exists an $x \in X$ such that $d_Y(f(x), y) \leq D$. The first condition is often called "quasi–isometry", and the second is often called "coarse surjectivity". Two finitely generated groups are quasi–isometric if their Cayley graphs are quasi–isometric metric spaces.

In the article under review, the coarse equivalence of countable metric spaces is stated as follows: X and Y are coarsely equivalent if there are maps $f : X \to Y$ and $g : Y \to X$ satisfying $d_X(g \circ f(x), x) \leq K$ and $d_Y(f \circ g(y), y) \leq K$ for all x and y. Furthermore, f and g must be bornologous, which is to say that they send balls of a given diameter into uniformly bounded balls. It is a basic result that coarse equivalence and quasi–isometry are equivalent.

The coarse classification of many classes of groups has been completed. In the present article, the authors carry out the classification for countable abelian groups. The classification carried out here only has content for infinitely generated groups, since finitely generated abelian groups are coarsely classified by their torsion–free rank.

The authors show that a group's torsion–free rank (namely the largest rank of a free abelian group embedded in a given group) and its asymptotic dimension determine its coarse equivalence class. Precisely, they show that for two countable abelian groups G and H endowed with proper left–invariant metrics, the following three statements are equivalent:

(1) G and H are coarsely equivalent.
(2) The asymptotic dimensions of G and H coincide and G and H are both large–scale connected or both not large–scale connected.
(3) The torsion–free ranks of G and H coincide and G and H are either both finitely generated or both infinitely generated.

The authors then determine which groups are coarsely equivalent to abelian groups.

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