Math table, February 24, 2009, Oliver Knill

Treasure Hunting Perfect Euler bricks

Abstract

An Euler brick is a cuboid with integer side dimensions such that the face diagonals are integers. Already in 1740, families of Euler bricks have been found. Euler himself constructed more families. If the space diagonal of an Euler brick is an integer too, an Euler brick is called a perfect Euler brick. Nobody has found one. There might be none. Nevertheless, it is an entertaining sport to go for this treasure hunt for rational cuboids and search - of course with the help of computers. We especially look in this lecture at Soundersons parametrizations and give a short proof of a theorem of Spohn telling that the any of these Euler bricks is not perfect. But there are other parameterizations.

Introduction: the map of John Flint

An **Euler brick** is a cuboid of integer side dimensions a, b, c such that the face diagonals are integers. If u, v, w are integers satisfying $u^2 + v^2 = w^2$, then the Sounderson parametrization

$$(a, b, c) = (|u(4v^2 - w^2)|, |v(4u^2 - w^2)|, |4uvw|)$$

leads to an Euler brick.

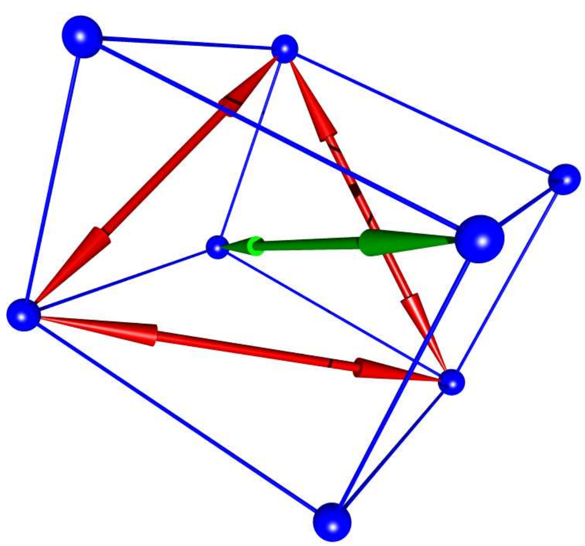

Fig 1. An Euler brick has integer face diagonals. It is perfect if the long diagonal is an integer too.

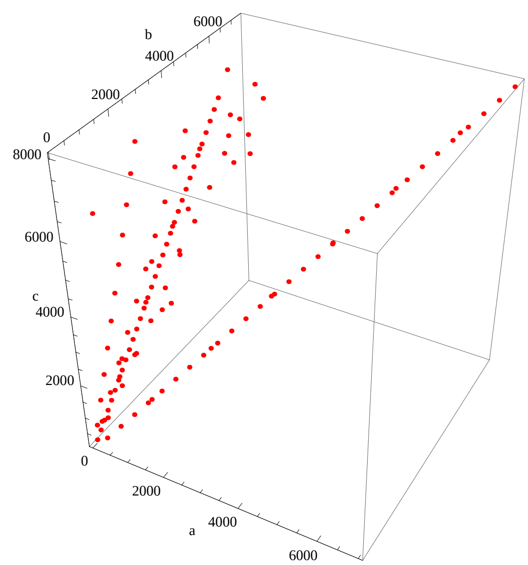

Fig 2. The smallest Euler bricks (a, b, c) with $a \leq b \leq c$ plotted in the parameter space.

The cuboid with dimensions $(a, b, c) = (240, 117, 44)$ is an example of an Euler brick. It is the smallest Euler brick. It has been found in 1719 by Paul Halcke (- 1731) [1].
If also the space diagonal is an integer, an Euler brick is called a **perfect Euler brick**. In other words, a cuboid has the properties that the vertex coordinates and all distances are integers.

It is an open mathematical problem, whether a perfect Euler bricks exist. Nobody has found one, nor proven that it can not exist. One has to find integers (a, b, c) such that

$$\sqrt{a^2+b^2}, \sqrt{a^2+c^2}, \sqrt{b^2+c^2}, \sqrt{a^2+b^2+c^2}$$

are integers. This is called a system of Diophantine equations. You can verify yourself that that the Sounderson parametrization produces Euler bricks.
If we parametrize the Pythagorean triples with $u = 2st, v = s^2 - t^2, w = s^2 + t^2$, we get $a = 6ts^5 - 20t^3s^3 + 6t^5s$, $b = -s^6 + 15t^2s^4 - 15t^4s^2 + t^6$, $c = 8s^5t - 8st^5$. This defines a parametrized surface

$$r(s,t) = \langle 6ts^5 - 20t^3s^3 + 6t^5s, -s^6 + 15t^2s^4 - 15t^4s^2 + t^6, 8s^5t - 8st^5\rangle$$

which leads for integer s, t to Euler bricks.

Indeed, one has then: $a^2+b^2 = (s^2+t^2)^6$, $a^2+c^2 = 4(5s^5t - 6s^3t^3 + 5st^5)^2$ $b^2+c^2 = (s^6 + 17s^4t^2 - 17s^2t^4 - t^6)^2$.
A perfect Euler brick would be obtained if $f(t,s) = a^2+b^2+c^2 = s^8 + 68 * s^6 * t^2 - 122 * s^4 * t^4 + 68 * s^2 * t^6 + t^8$ were a square.

Brute force search: yo-ho-ho and a bottle of rum!

There are many Euler bricks which is not parametrized as above:
A brute force search for $1 \leq a, b, c \leq 300$ gives $a = 44, b = 117, c = 240$ and $a = 240, b = 252 c = 275$ as the only two Euler bricks in that range. In the range $1 \leq a < b < c \leq 1000$ there are 10 Euler bricks:

a	b	c
44	117	240
85	132	720
88	234	480
132	351	720
140	480	693
160	231	792
176	468	960
240	252	275
480	504	550
720	756	825

In the $1 \leq a < b < c \leq 2000$, there are a 15 more, totalling 25.

a	b	c
170	264	1440
187	1020	1584
220	585	1200
264	702	1440
280	960	1386
308	819	1680
320	462	1584
352	936	1920
480	504	550
720	756	825
960	1008	1100
1008	1100	1155
1200	1260	1375
1440	1512	1650
1680	1764	1925

Searching $1 \leq a < b < c \leq 4000$, we get 54 Euler cuboids, in $1 \leq a < b < c \leq 8000$ there are 120:

44	117	240	528	5796	6325	968	2574	5280	1680	1764	1925
85	132	720	560	1920	2772	980	3360	4851	1755	4576	6732
88	234	480	561	3060	4752	1008	1100	1155	1920	2016	2200
132	351	720	572	1521	3120	1012	2691	5520	2016	2200	2310
140	480	693	595	924	5040	1056	2808	5760	2160	2268	2475
160	231	792	616	1638	3360	1100	2925	6000	2400	2520	2750
170	264	1440	640	924	3168	1120	1617	5544	2496	2565	7920
176	468	960	660	1755	3600	1120	3840	5544	2640	2772	3025
187	1020	1584	680	1056	5760	1144	3042	6240	2880	3024	3300
195	748	6336	700	2400	3465	1155	6300	6688	3024	3300	3465
220	585	1200	704	1872	3840	1188	3159	6480	3120	3276	3575
240	252	275	720	756	825	1200	1260	1375	3360	3528	3850
255	396	2160	748	1989	4080	1232	3276	6720	3600	3780	4125
264	702	1440	748	4080	6336	1260	4320	6237	3840	4032	4400
280	960	1386	765	1188	6480	1276	3393	6960	4032	4400	4620
308	819	1680	780	2475	2992	1280	1848	6336	4080	4284	4675
320	462	1584	792	2106	4320	1287	2640	7020	4320	4536	4950
340	528	2880	800	1155	3960	1320	3510	7200	4560	4788	5225
352	936	1920	828	2035	3120	1364	3627	7440	4800	5040	5500
374	2040	3168	832	855	2640	1400	4800	6930	5040	5292	5775
396	1053	2160	836	2223	4560	1408	3744	7680	5040	5500	5775
420	1440	2079	840	2880	4158	1440	1512	1650	5280	5544	6050
425	660	3600	850	1320	7200	1440	2079	7128	5520	5796	6325
429	880	2340	858	1760	4680	1452	3861	7920	5760	6048	6600
440	1170	2400	880	2340	4800	1540	5280	7623	6000	6300	6875
480	504	550	924	2457	5040	1560	2295	5984	6048	6600	6930
480	693	2376	935	1452	7920	1560	4950	5984	6240	6552	7150
484	1287	2640	935	5100	7920	1600	2310	7920	6480	6804	7425
510	792	4320	960	1008	1100	1656	4070	6240	6720	7056	7700
528	1404	2880	960	1386	4752	1664	1710	5280	6960	7308	7975

The number of Euler bricks appears to grow linearly with respect to the box because if (a, b, c) is an Euler brick, then (ka, kb, kc) is an Euler brick too. It would be interesting to know how primitive Euler bricks are distributed.

Modular considerations: pieces of eight! pieces of eight!

If we take a Diophantine equation and consider it modulo some number n, then the equation still holds. Turning things around: if a Diophantine equation has no solution modulo n, then there is no solution in the integers. By checking all possible solutions in the finite space of all possible cases, we can also determine some conditions, which have to hold.

Example: $5x^4 = 3 + 7y^4$ has no integer solutions because modulo 8, we have no solution because modulo 8 we have $x^4, y^4 \in \{0, 1\}$.

To use this idea, lets assume we deal with prime Euler bricks, bricks for which the greatest common divisor of a, b, c is 1.

For $n \in \{2, 3, 5, 11\}$ as well as $n \in \{2^2, 3^2, 4^2\}$, there exists at least one side of an Euler brick which is divisible by n.

Proof. The case $n = 2, 4, 16$ follows directly from properties of Pythagorean triples, for $n = 9$, use that if two (say x, y) are divisible by 3, then $x^2 - y^2 = a^2 - b^2$ is divisible by 9 and $a = b$ (mod 3) showing that also z has to be divisible by 3 and the cube is not prime.

Searching using irrational rotation: on a dead man's chest

The problem of solving Diophantine equations has a dynamical system side to it. Take one of the variables x as time, solve with respect to an other variable say y then write $y = (f(x))^{1/n}$ where f is a polynomial. We can study the dynamical system $(f(x))^{1/k} \to f(x+1)^{1/k}$ mod 1 and look for n to reach 0. If there are several parameters, we have a dynamical system with multidimensional time.

For the problem to find s, t for which

$$\sqrt{s^8 + 68t^2s^6 - 122t^4s^4 + 68t^6s^2 + t^8}$$

is close to an integer, we can change the parameter s, t along a line and get incredibly close. Unfortunately, we can not hit a lattice point.

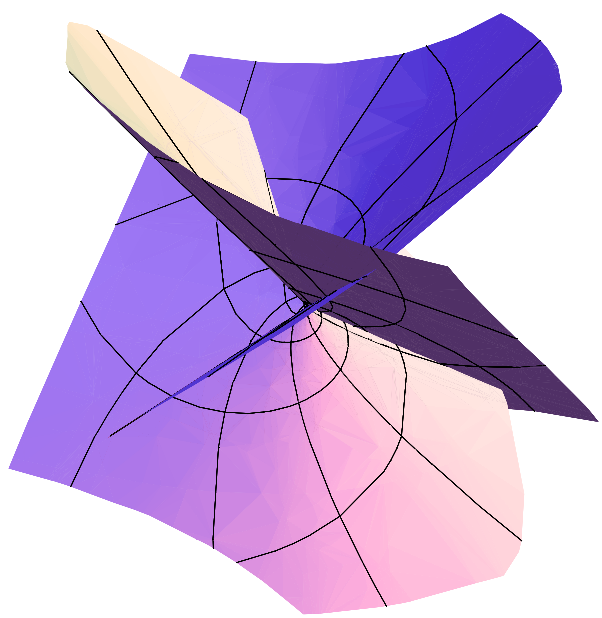

Fig 1. The Sounderson surface: a parametrized surface $r(s, t) = (a(s, t), b(s, t), c(s, t))$ of Euler bricks.

The treasure is not there: ney mate, you are marooned

Spohn is the "Ben Gunn" of the Euler brick treasure island. He has moved the treasure elsewhere. But maybe it does not exist. Anyway, Spohn [10] proved in 1972:

Theorem (Spohn): There are no perfect Euler bricks on the Sounderson surface of Euler bricks.

Proof. With $a = u(4v^2 - w^2)$, $b = v(4u^2 - w^2)$, $c = 4uvw; w^2 = u^2 + v^2$, we check $a^2 + b^2 + c^2 = w^2(u^4 + 18u^2v^2 + v^4)$. Pocklington [8] has shown first in 1912 that $u^4 + 18u^2v^2 + v^4$ can not be be square. His argument is more general. We can prove this more easily however:

Lemma (Pocklington): Unless $xy = 0$, the Diophantine equation $x^4 + 18x^2y^2 + y^4 = z^2$ has no solution.

Proof: x, y can not have a common factor, otherwise we could divide it out and include it to z. Especially, there is no common factor 2. If $x^4 + 18x^2y^2 + y^4 = (x^2 + y^2)^2 + 4^2x^2y^2 = z^2$ then we have Pythagorean triples which can be parametrized.
a) Assume first the triples are primitive, there is no common divisor among the triple $(x^2 + y^2)^2, 4^2x^2y^2, z^2)$.

(i) If x, y are both odd, we must have

$$\begin{aligned} x^2 + y^2 &= 2uv \\ 4xy &= u^2 - v^2 \; . \end{aligned}$$

The first equation proves that $x^2 + y^2 = 2 \bmod 4$. If $2uv = 2 \bmod 4$, both u, v must be odd. The second equation can now not be solved modulo 8. If $u = 4n \pm 1$, $v = 4m \pm 1$, then $u^2 - v^2$ is divisible by 8. But the left hand side of the equation is congruent to 4 modulo 8.
(ii) If x is odd and y is even, the Pythagorean triple representation is

$$\begin{aligned} x^2 + y^2 &= u^2 - v^2 \\ 2xy &= uv \; . \end{aligned}$$

Because y is even, the second equation shows that uv is divisible by 4 and because u, v have no common divisor, wither u is divisible by 4 or v is divisible by 4. If u is divisible by 4, the first equation can not be solved modulo 4. If v is divisible by 4, the first equation has no solution modulo 16: the right hand side is $0, 1, 4, 9$ modulo 16 while the left hand side is congruent to $5, 13$ modulo 16.

b) If there is a common divisor p among $(x^2 + y^2)^2$ and $4^2x^2y^2$ then it has to be 2, because any other factor p would be a factor of either x or y as well as of $x^2 + y^2$ and so of both x and y, which we had excluded at the very beginning. With a common factor 2, we have a Pythagorean triple parametrization

$$\begin{aligned} x^2 + y^2 &= 4uv \\ 4xy &= 2(u^2 - v^2) \; . \end{aligned}$$

but since x, y are both odd, $x^2 + y^2$ is congruent 2 modulo 4 contradicting the first equation.

This finishes the proof of the lemma and so the theorem of Spohn. It is remarkable that the result of Pocklington does not use infinite decent in this case. By the way, the article of Pocklington of 1912 has been checked out many times at Cabot library since this volume almost falls to dust. Side remark: quartic Diophantine equations of this type form an old topic [6] (section 4). Fermat had shown using infinite descent that u^4+v^4 is never a square so that $u^4+v^4 = z^4$ has no solution. As is well known, he concluded a bit hastily that he has a proof that $x^p + y^p = z^p$ has no solution for all $p > 2$ but that the margin is not large enough to hold it.

Large numbers: shiver my timbers!

There are more parametrizations to be explored. Euler got

$$\begin{aligned} a &= 2mn(3m^2 - n^2)(3n^2 - m^2) \\ b &= 8mn(m^4 - n^4) \\ c &= (m^2 - n^2)(m^2 - 4mn + n^2)(m^2 + 4mn + n^2) \end{aligned}$$

for which $x^2 + y^2 = 4m^2n^2(5m^4 - 6m^2n^2 + 5n^4)^2$, $^2 + z^2 = (m^2 + n^2)^6, y^2 + z^2 = (m - n)^2(m + n)^2(m^4 + 18m^2n^2 + n^4)^2$.
In that case, we have $x^2 + y^2 + z^2 = (m^2 + n^2)^2(m^8 + 68m^6n^2 - 122m^4n^4 + 68m^2n^6 + n^8) = (m^2 + n^2)^2[(m^4 + n^4)^2 + 2m^2n^2(17m^4 - 31m^2n^2 + 17n^4)]$.

Computer algebra systems like to compute as long as possible in algebraic fields. For example:

```
Expand[(5 + Sqrt[5])^6]
```

produces the result

```
72000+32000 Sqrt[5]
```

This is a much more valuable result than a numerical value like 143554.1753.... The evaluation of numerical values in Mathematica is quite mysterious: sometimes, it works quite well:

```
N[Sqrt[2^171 + 1]] - N[Sqrt[2^171 + 1], 100]
```

Sometimes, it does not

```
N[Sqrt[2^117 + 1]] - N[Sqrt[2^117 + 1], 100]
```

which gives in this case a value of -64. Even increasing the accuracy like with

```
$MaxExtraPrecision=20000000000;
```

Wolfram research promised to fix this problem.

By the way, this issue is much better in Pari. How to compute with large accuracy in the open source algebra system Pari/GP? Pari projects algebraic integers correctly, even with millions of digits:

```
\p 1000000
a=sqrt(2^117+1)
```

It can compute up to 161 million significant digits (you have to increase the stack size to do so), like defining

```
parisize = 800M
```

in the .gprc file. It still can produce an overflow depending on your machine. But working a million digits or so is ok.

History: Captain Flints logbook

In 1719 by **Paul Halcke**, a German accountant, who would also do astronomical computations, found the smallest solution [1]. Nothing earlier seems to be known.

N. Sounderson found in 1740 the parametrization with two parameters mentioned above. Only in 1972, it was established by Spohn that the parametrization does not lead to that these parametrizations do not lead to perfect Euler bricks. Jean Lagrange gave an other argument in 1979 also.

Leonard Euler found in 1770 a second parametrization and in 1772 a third parametrization. After his death, more parametrizations were found in his notes.

Modern considerations: the black spot

The topic has appeared several times in American Mathematical Monthly articles and was even a topic for a PhD theses in 2000 in Europe and 2004 in China. Because of its simplicity, it is certainly of great educational value. The topic appears for example in a journal run by undergraduates similar to HCMR [7].

Noam Elkies told me:

"The alebraic surface parametrizing Euler bricks is the intersection in P^5 of the quadrics $x^2+y^2=c^2$, $z^2+x^2=b^2$, $y^2+z^2=a^2$ which happens to be a K3 surface of maximal rank, so quite closely related to much of my own recent work in number theory. Adding the condition $x^2+y^2+z^2=d^2$ yields a surface of general type, so it might well have no nontrivial rational points but nobody knows how to prove such a thing."

Noam also remarked that Eulers parametrization would only lead to a finite number of perfect Cuboids as a consequence of Mordells theorem. There seems however no reason to be known which would tell whether there are maximally finitely many primitive perfect cuboids.

There are also relations with elliptic curves since a system of quadratic equations often define an elliptic curve. See [5]. The article [4] which mentions also relations with rational points on plane cubic curves.

The problem appeared also in articles for the general public. In 1970 Martin Gardner asked to find solutions for which 6 of the 7 distances in the cuboid are integers. If the large diagonal is an integer, these are no more Euler bricks, unless we would have a perfect brick.

As for any open problem, it is also interesting to look more fundamental questions. As with many open problems, the problem to find a perfect Euler brick could be undecidable: we would not be able to find a proof that there exists no Euler brick. This is possible only if there is indeed no Euler brick. You can read an amusing story about Goldbach conjecture in "Uncle Petros and the Goldbach conjecture", where the perspective of such an option blew all motivation of poor uncle Petros to search for the Goldbach grail. [2]

Treasure problems: scatter and find 'em!

Many unsolved problems like the Goldbach conjecture, the Riemann hypothesis, the problem to find perfect numbers, or the problem of finding perfect Euler bricks, finding dense sphere packings in higher dimensions, are mathematical tasks which could in principle be solved quickly: by finding an example - if it should exist:

- Writing down an integer which can not be written as a sum of two primes would settle the Goldbach conjecture.
- Finding an integer for which the sum of the proper divisors is the number itself.
- Find a root of the zeta function with $Re(z) \neq 1/2$. Just one lucky punch would be needed to solve the problem.

But like treasure hunting, aiming to catch such a treasure is not a good business plan or a way to make a living: the treasure simply does not need to be there. If it does not exist, the most skillful treasure hunter can not be successful.

But it is the search which is interesting, not the prospect of finding anything.

By the way, numerical searches for the grail of a perfect cuboid have been done. Randal Rathbun has found no perfect cuboid with least edge larger than 333750000. The greatest edge is larger than 10^9. See [3]. Treasure hunters all over the world have probably gone even further. See [9] on arxiv.

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