8/3/2017 PRACTICE EXAM VI **Maths 21a, O. Knill, Summer 2017**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

1) T F The lines $\vec{r_1}(t) = \langle t, t, -t \rangle$ and $\vec{r_2}(t) = \langle 1+t, 1+t, 1-t \rangle$ do not intersect.

Solution:
Indeed, they are parallel.

2) T F The quadratic surface $x^2 - y^2 = z^2$ is a hyperbolic paraboloid.

Solution:
No, it is a cone.

3) T F If $\vec{T}(t), \vec{B}(t), \vec{N}(t)$ are the unit tangent, normal and binormal vectors of a curve with $\vec{r}\,'(t) \neq 0$ everywhere, then they span a parallelepiped of volume 1.

Solution:
The three vectors are perpendicular and have length 1.The parallelepiped is a cube of volume 1.

4) T F If $\vec{u} \cdot \vec{v} = 0$, then $\text{Proj}_{\vec{v}}(\vec{u}) = \vec{0}$.

Solution:
The two vectors are then perpendicular.

5) T F There is a vector field $\vec{F}(x, y)$ which has the property $\text{curl}(\vec{F}) = -\text{div}(\vec{F})$, where $\text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y)$ and $\text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y)$.

Solution:
Take an incompressible irrotational field. Any constant field works

6) T F The acceleration vector $\vec{r}\,''(t) = \langle x(t), y(t), z(t) \rangle$ is always a unit vector if the velocity vector $\vec{r}\,'(t)$ is a unit vector.

Solution:
Take $\vec{r}(t) = \langle t^3/3, 0, 0 \rangle$ at time $t = 1$ for example

7) | T | F | (T highlighted)

The grid curves $t \to \vec{r}(t, \phi)$ with fixed $0 < \phi < \pi$ for the standard parametrization of the unit sphere have curvature $1/\sin(\phi)$.

Solution:
The radius is $\sin(\phi)$.

8) | T | F | (F highlighted)

Any smooth function $f(x, y)$ has a local maximum somewhere in the plane.

Solution:
$x^2 - y^2$ does not have a maximum, nor a minimum.

9) | T | F | (T highlighted)

The linearization $L(x, y)$ of constant function $f(x, y) = 3$ is $L(x, y) = 3$.

Solution:
The linearization of linear function is the same function

10) | T | F | (F highlighted)

A gradient field is incompressible: it satisfies $\text{div}(F) = 0$ everywhere.

Solution:
It is irrotational not necessarily incompressible. An example is $\vec{F}(x, y, z) = \langle x, y, z \rangle$.

11) | T | F | (F highlighted)

If $f(x, y)$ has a maximum under the constraint $g(x, y) = 1$, then $\nabla f = \langle 0, 0 \rangle$ at this point.

Solution:
Critical points under constraints are not necessarily critical points without constraint.

12) | T | F | (T highlighted)

Assume a vector field $\vec{F}(x, y, z)$ is the curl of a vector field $\vec{G}$ then the flux of the field F through the ellipsoid $x^2 + y^2 + 5z^2 \leq 1$ is zero.

Solution:
This follows both from Stokes theorem (no boundary curve) or the divergence theorem because $\text{div}(F) = \text{divcurl})(F) = 0$.

13) | T | F (highlighted) |

If the divergence of a field $\vec{F}$ are zero everywhere, then any line integral along a closed curve is zero.

Solution:
It is the curl which matters, when we look at line integrals along closed loops.

14) | T | F (highlighted) |

The gradient of the divergence of a field is always the zero field.

Solution:
Take $f(x,y,z) = \langle x^2, y^2, z^2 \rangle$ for which the gradient of the divergence is $\langle 2,2,2 \rangle$

15) | T (highlighted) | F |

The vector field $\vec{F}(x,y,z) = \langle x^2, y^2, z^3 \rangle$ is a gradient field.

Solution:
Yes, the potential is $f(x,y,z) = x^3/3 + y^3/3 + z^4/4$.

16) | T (highlighted) | F |

The volume of a solid can be computed as the flux of the field $\langle 0, y, 0 \rangle$ through the boundary surface.

Solution:
The field has constant divergence 1 so that by the divergence theorem the flux of the field through the boundary surface is equal to the volume.

17) | T (highlighted) | F |

The curvature of a line is zero.

Solution:
Yes, there is no change of the unit tangent vector. You can also see it from the fact that the velocity and acceleration are parallel. —

18) T F The distance between the unit sphere centered at $(0, 0, 0)$ and the plane $z = 5$ is equal to 4.

Solution:
It is by 1 smaller than the distance of the origin to the plane.

19) T F The partial differential equation $u_t = u_x$ is called heat equation.

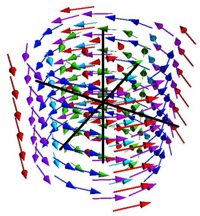
Solution:
It is the transport equation

20) T F The point $(1, -1, \sqrt{2})$ in spherical coordinates is $(\rho, \phi, \theta) = (2, \pi/4, 3\pi/2)$.

Solution:
The first two entries are correct but the θ angle does not match. It would be $7\pi/4$.

Problem 2) (10 points) No justifications are necessary.

a) (4 points) Match the objects with the definitions.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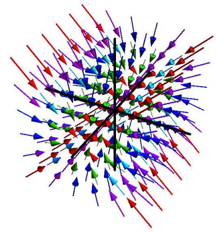
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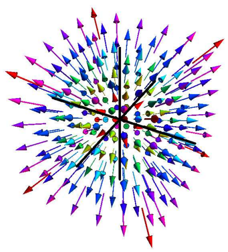
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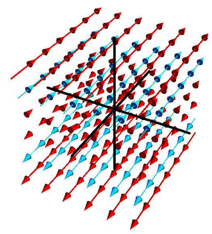
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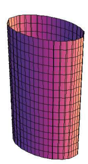
enter 1-4	vector field
	$\vec{F}(x, y, z) = \langle x, y, z \rangle$
	$\vec{F}(x, y, z) = \langle -y, x, 0 \rangle$
	$\vec{F}(x, y, z) = \langle 0, z, 0 \rangle$
	$\vec{F}(x, y, z) = \langle -x, 0, -z \rangle$

Solution:

b) (3 points) Match the surfaces with their names: (put O if no match)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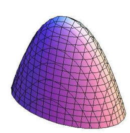
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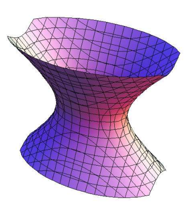
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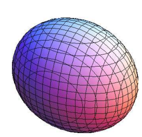
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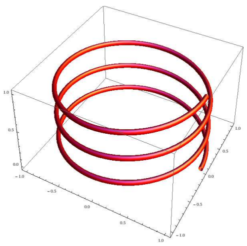
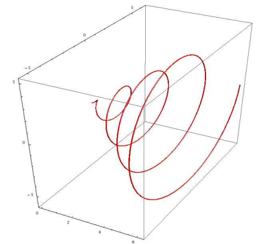
enter 1-4	surface
	$x^2 + y^2 + 3z = 0$
	$x^2 + y^2 - 3z^2 = 1$
	$x^2 + y^2 + 3z^2 = 1$
	$x^3 + 3y^2 = 1$

c) (3 points) Match the space curves



1



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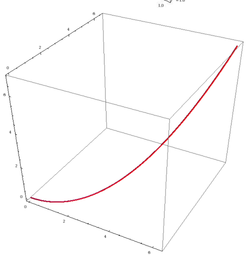

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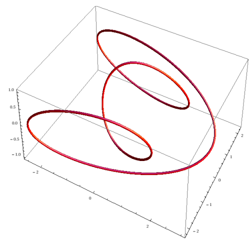


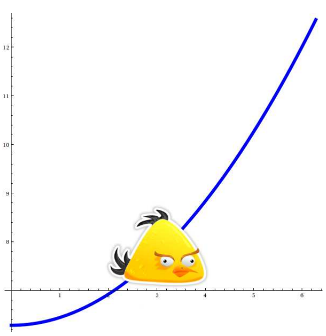
4

1-4	parametrized curve
	$\vec{r}(t) = \langle t, t^2, t^3 \rangle$
	$\vec{r}(t) = \langle \cos(3t), \sin(3t), t \rangle$
	$\vec{r}(t) = \langle (2+\cos(t))\cos(3t),$ $(2+\cos(t))\sin(t),$ $\sin(3t) \rangle$
	$\vec{r}(t) = \langle t, t\cos(3t), t\sin(3t) \rangle$

Solution:
a) 3,1,4,2
b) 2,3,4 0, The $x^3 + 3y^2 = 1$ is cylindrical but but not bounded in the xy plane.
c) 3,1,4,2

Problem 3) (10 points) No justifications are necessary

a) (5 points) We watch "angry birds" attacking on curves with acceleration $\vec{r}\,''(t)$. (The pictures show the $xz-$ planes and the birds start with a constant velocity $\langle 1, 0, 0 \rangle$.) Match the displayed curves $\vec{r}(t)$ with the formulas for accelerations.



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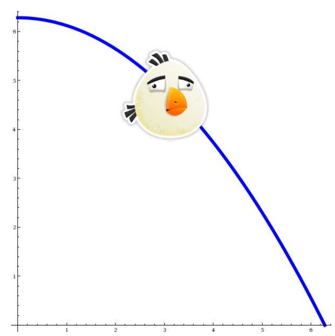

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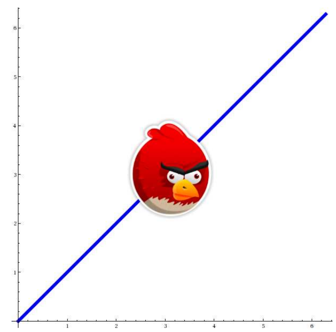
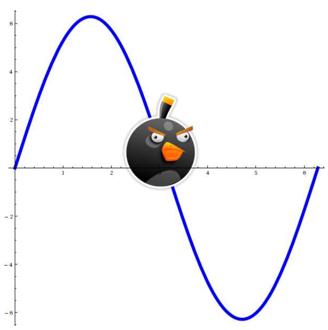


acceleration	enter curve 1-5
$\vec{r}\,''(t) = \langle 0, 0, \sin(t) \rangle$	
$\vec{r}\,''(t) = \langle 0, 0, -10 \rangle$	
$\vec{r}\,''(t) = \langle 0, 0, 10 \rangle$	
$\vec{r}\,''(t) = \langle 0, 0, -\sin(t) \rangle$	
$\vec{r}\,''(t) = \langle 0, 0, 0 \rangle$	

3



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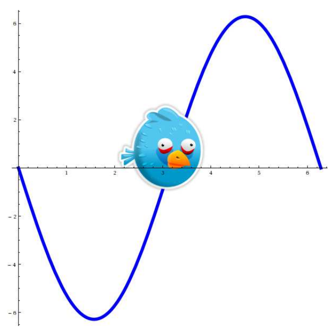

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b) (5 points) Match the formulas: (put O if no match)

label	formula
A	$\vec{r}\,'(t)$
B	$\int_0^1 \lvert\vec{r}\,'(t)\rvert \, dt$
C	$\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$
D	$\vec{T}\,'(t)/\lvert\vec{T}\,'(t)\rvert$
E	$\lvert\vec{r}'(t) \times \vec{r}''(t)\rvert/\lvert\vec{r}'(t)\rvert^3$

expression	enter A-E
Curvature	
Unit tangent vector	
Unit normal vector	
Velocity	
Arc length	

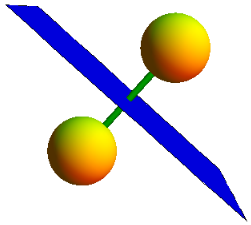
Solution:

a) 5,2,1,4,3

A difficulty had been the curves with the $\sin(t)$ entries. Integrating this twice gives $-\sin(t)$. The $\sin(t)$ z-acceleration produces the curve $r(t)$ with $-\sin(t)$ in the z coordinate.

b) E,C,D,A,B.

Problem 4) (10 points)

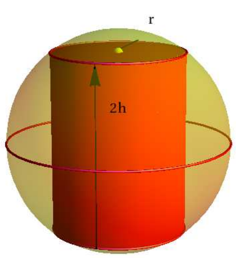


a) (5 points) Find a parametrization of the line L through the center of the two spheres $x^2 + (y-1)^2 + z^2 = 1$, $(x-5)^2 + y^2 + z^2 = 1$.

b) (5 points) Find the plane perpendicular to the line L for which the distances to the spheres are the same.

Solution:
a) The centers of the sphere are $(0,1,0)$ and $(5,0,0)$. A parametrization of the line connecting these two points is is $\vec{r}(t) = \langle 0,1,0\rangle + t\langle 5,-1,0\rangle$. It can be written also as $\boxed{\vec{r}(t) = \langle 5t, 1-t, 0\rangle}$.
b) The midpoint between the two points is $(5/2, 1/2, 0)$. Because the the normal vector to the plane is $\langle 5,-1,0\rangle$, the plane has the equation $5x - y = d$ where the constant d can be obtained by plugging in the midpoint. The answer is $\boxed{5x - y = 12}$.

Problem 5) (10 points)



Johannes Kepler asked which cylinder or radius r and height $2h$ inscribed in the unit sphere has maximal volume. To solve his problem, use the Lagrange method and maximize the volume

$$f = 2\pi r^2 h$$

under the constraint that $r^2 + h^2 = 1$.

Solution:
We have to extremize

$$f(x, y) = 2\pi x^2 y$$

under the constraint

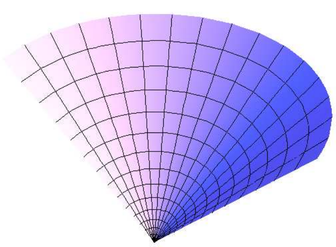
$$g(x, y) = x^2 + y^2 = 1 \ .$$

The Lagrange equations are

$$\begin{aligned} 4\pi rh &= \lambda 2r \\ 2\pi r^2 &= \lambda 2h \\ r^2 + h^2 &= 1 \end{aligned}$$

Divide the first by the second to get $2h/r = r/h$ or $2h^2 = r^2$ which gives $3h^2 = 1$. or $\boxed{h = 1/\sqrt{3}}$ and $\boxed{r = \sqrt{2}/\sqrt{3}}$.

Problem 6) (10 points)



a) (6 points) Find the surface area of the surface

$$r(u, v) = \langle v^2\cos(u), v^2\sin(u), v^2\rangle, 0 \le u \le \pi, 0 \le v \le 1 \ .$$

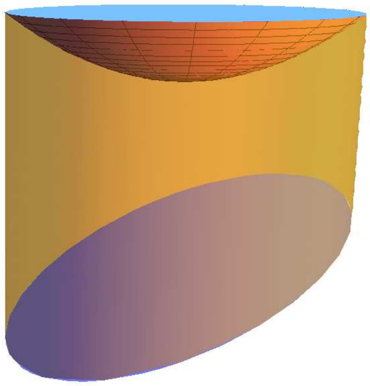
b) (4 points) Find the arc length of the boundary curve $\vec{r}(u, 1)$ where $0 \le u \le \pi$.

Solution:
a) $\vec{r}_u = \langle -v^2\sin(u), v^2\cos(u), 0\rangle$.
$\vec{r}_v = \langle 2v\cos(u), 2v\sin(u), 2v\rangle$.
The cross product is $2v^3\langle\cos(u), \sin(u), -1\rangle$ which has length $\sqrt{2}2v^3$. Integrating this over the parameter domain gives

$$\int_0^\pi \int_0^1 v^3\sqrt{2}2 \ dvdu = \frac{v^4}{4}|_0^1\sqrt{2}2 = \pi\sqrt{2}/2 \ .$$

This is also $\boxed{\pi/\sqrt{2}}$.
b) The curve $\vec{r}(u) = \langle\cos(u), \sin(u), 0\rangle$ has velocity $\vec{r}\,'(u) = \langle -\sin(u), \cos(u), 0\rangle$ which has length 1. The arc length is $\boxed{\pi}$.

Problem 7) (10 points)



Find the volume of the solid inside the cylinder

$$x^2 + y^2 \leq 2$$

sandwiched between the graphs of $f(x,y) = x - y$ and $g(x,y) = x^2 + y^2 + 4$.

Solution:
The best setup is in **cylindrical coordinates**, where $x - y = r\cos(\theta) - r\sin(\theta)$ and $x^2 + y^2 + 4 = r^2 + 4$:

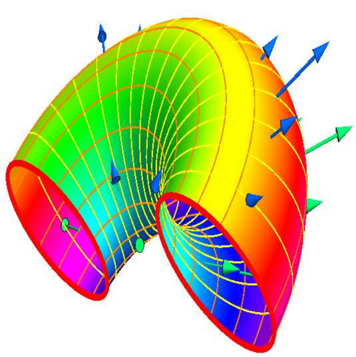
$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r\cos(\theta)-r\sin(\theta)}^{r^2+4} r \; dz dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} r^2 + 4 - r\cos(\theta) - r\sin(\theta) \; r dr \; d\theta \; .$$

Since integrating $\sin(\theta)$ and $\cos(\theta)$ from 0 to 2π is zero, we only have

$$\int_0^{2\pi} \int_0^{\sqrt{2}} (r^3 + 4r) \; dr \; d\theta = 2\pi(r^4/4 + 4r^2/2)|_0^{\sqrt{2}} = 2\pi \cdot 5 \; .$$

The answer is $\boxed{10\pi}$.

Problem 8) (10 points)



Find the flux of the curl of the vector field

$$\vec{F}(x,y,z) = \langle x, y, z + \sin(\sin(y^2)) \rangle$$

through the torus

$$\vec{r}(s,t) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s) \rangle$$

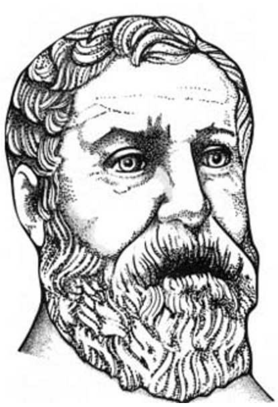
with $0 \leq t \leq \pi$ and $0 \leq s < 2\pi$.

Solution:
This is a problem for Stokes theorem. We can find the boundary curves by setting $t = 0$ or $t = \pi$. The two curves are

$$C_1 : r_1(s) = \langle (2 + \cos(s)), 0, \sin(s)\rangle, C_2 : r_2(s) = \langle -2 - \cos(s), 0, \sin(s)$$

both parametrized for $0 \leq s \leq 2\pi$. The flux is the sum of the line integral of $\vec{F}$ along these two curves: $\int_{C_1} \vec{F}\ \vec{ds} + \int_{C_2} \vec{F}\ \vec{ds} = \int_0^{2\pi} \langle 2 + \cos(s), 0, \sin(s)\rangle \cdot \langle -\sin(s), 0, \cos(s)\rangle\ ds + \int_0^{2\pi} \langle -2 - \cos(s), 0, \sin(s)\rangle \cdot \langle \sin(s), 0, \cos(s)\rangle\ ds = 0 + 0\ = 0$. The answer is $\boxed{0}$.

Problem 9) (10 points)



Heron's formula for the area A of a triangle of side length $x, y, 1$ satisfies $16A^2 = f(x, y)$, where

$$f(x, y) = -1 + 2x^2 - x^4 + 2y^2 + 2x^2y^2 - y^4\ .$$

Classify all the critical points of f. Is there a global maximum of f and so for the area?

Remark not to worry about: The formula follows directly from Heron's formula $s = (a + b + 1)/2; A = \sqrt{s(s-a)(s-b)(s-1)}$.

Solution:
The critical points satisfy the equations $\nabla f(x,y) = \langle 0,0 \rangle$ which is

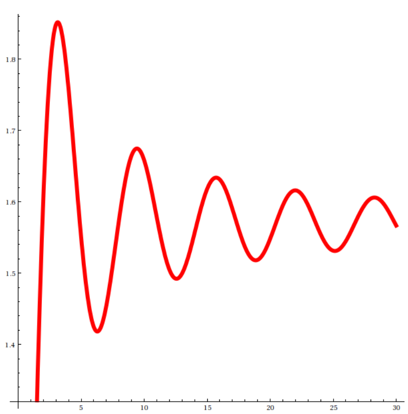
$$\begin{aligned} 4x(1-x^2+y^2) &= 0 \\ 4y(1+x^2-y^2) &= 0 \ . \end{aligned}$$

To satisfy this there are four possibilities: either $x=0, y=0$ or $x=0, 1+x^2-y^2=0$ or $1-x^2+y^2=0, y=0$ or $1-x^2+y^2=0, 1+x^2-y^2=0$. This is $(0,0), (0,\pm 1), (\pm 1, 0)$. We have $D(x,y) = 8xy$ and $f_{xx} = 4(1-3x^2+y^2)$.

x	y	D	f_{xx}	classification	f
-1	0	-64	-8	saddle	0
0	-1	-64	8	saddle	0
0	0	16	4	minimum	-1
0	1	-64	8	saddle	0
1	0	-64	-8	saddle	0

There is no global maximum because $f(x,x) = 4x^2 - 1$ grows to infinity. There is no limit on areas of triangles we can build. $f(x,x)/16$ is the square of the area of an isoscele triangle with side lengths $1, x, x$.

Problem 10) (10 points)



The anti derivative of the **sinc** function

$$\frac{\sin(x)}{x}$$

is called the **sine integral** $\mathrm{Si}(x)$. It can not be expressed in terms of known functions. Still we can compute the following double integral

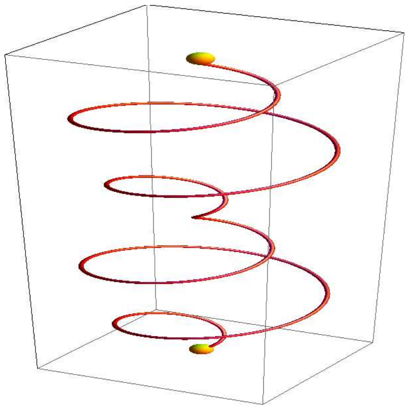
$$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y} \, dy \, dx \ .$$

Solution:
Change the order of integration

$$\int_0^\pi \int_0^y \frac{\sin(y)}{y} \, dx \, dy = \int_0^\pi \sin(y) \, dy = -\cos(y)|_0^\pi = 2 \ .$$

The answer is $\boxed{2}$.

Problem 11) (10 points)



Find the line integral of the vector field

$$\vec{F}(x,y,z) = \langle -x^{10}, \sin(y), z^3 \rangle$$

along the curve $\vec{r}(t) = \langle \sin(t)\cos(5t), \sin(t)\sin(5t), t \rangle$ where $0 \leq t \leq 2\pi$.

Solution:
The vector field is a gradient field $\vec{F} = \nabla f$ with

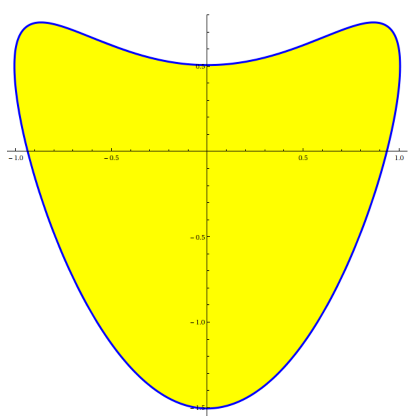
$$f(x,y) = -x^{11}/11 - \cos(y) + z^4/4 \ .$$

The curve end points are $f(r(2\pi)) = f(0,0,2\pi)$ and $f(r(0)) = f((0,0,0))$. The fundamental theorem of line integrals assures that

$$\int_C \vec{F}\ d\vec{r} = (2\pi)^4/4 - 0$$

which is $\boxed{4\pi^4}$.

Problem 12) (10 points)



Find the area of the region enclosed by the curve

$$\vec{r}(t) = \langle \cos(t), \sin(t) + \cos(2t)/2 \rangle \ ,$$

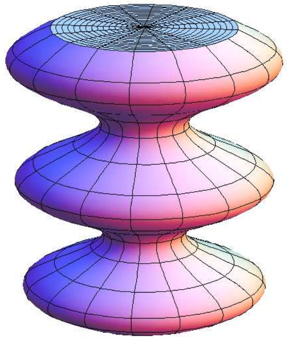
where $0 \leq t < 2\pi$.

Solution:
Use the vector field $\vec{F}(x,y) = \langle 0, x \rangle$ which has constant curl 1. We can compute the area by computing the line integral of $\vec{F}$ along the boundary. This is

$$\int_0^{2\pi} \langle 0, \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) - \sin(2t) \rangle \, dt = \int_0^{2\pi} \cos^2(t) - 2\cos^2(t)\sin(t) \, dt = \int_0^{2\pi} \cos^2(t) \, dt = \pi \, .$$

The answer is $\boxed{\pi}$.

Problem 13) (10 points)



Find the flux of the vector field

$$\vec{F}(x,y,z) = \langle x^3/3, y^3/3, \sin(xy^5) \rangle$$

through the boundary surface of the solid bound by the surface of revolution $\vec{r}(t,z) = \langle (2+\sin(z))\cos(t), (2+\sin(z))\sin(t), z \rangle$ and the planes $z = 0$, $z = 3$. The surface is oriented so that the normal vector points outwards.

Solution:
We use the divergence theorem. Since $\text{div}(\vec{F}) = x^2 + y^2 = r^2$, we have to integrate this over the solid

$$\int_0^{2\pi} \int_0^3 \int_0^{2+\sin(z)} r^2 \cdot r \, dr \, dz \, d\theta = (2\pi) \int_0^3 \frac{r^4}{4} \Big|_0^{2+\sin(z)} \, dz = \frac{\pi}{2} \int_0^3 (2+\sin(z))^4 \, dz \, .$$

It was sufficient to reach this integral $\boxed{\frac{\pi}{2} \int_0^3 (2+\sin(z))^4 \, dz}$ to get full credit. The actual answer is $(\pi/64)(11756 - 3648\cos(3) + 64\cos(9) - 600\sin(6) + 3\sin(12))$.