

Lecture 10: Linearization

In single variable calculus, you have seen the following definition:

The linear approximation of \( f(x) \) at a point \( a \) is the linear function

\[
L(x) = f(a) + f'(a)(x-a) .
\]

The graph of the function \( L \) is close to the graph of \( f \) at \( a \). We generalize this now to higher dimensions:

The linear approximation of \( f(x, y) \) at \((a, b)\) is the linear function

\[
L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) .
\]

The linear approximation of a function \( f(x, y, z) \) at \((a, b, c)\) is

\[
L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x-a) + f_y(a, b, c)(y-b) + f_z(a, b, c)(z-c) .
\]

Using the gradient

\[
\nabla f(x, y) = (f_x, f_y), \quad \nabla f(x, y, z) = (f_x, f_y, f_z) ,
\]

the linearization can be written more compactly as

\[
L(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) .
\]

How do we justify the linearization? If the second variable \( b = \) fixed, we have a one-dimensional situation, where the only variable is \( x \). Now \( f(x, b) = f(a, b)(x-a) \) is the linear approximation. Similarly, if \( x = x_0 \) is fixed \( y \) is the single variable, then \( f(x_0, y) = f(x_0, y_0) + f_y(x_0, y_0)(y-y_0) \). Knowing the linear approximations in both \( x \) and \( y \) variables, we can get the general linear approximation by \( f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) \).

1. What is the linear approximation of the function \( f(x, y) = \sin(\pi xy^2) \) at the point \((1, 1)\)? We have \( f(x, y, y_0) = (\pi y^2 \cos(\pi xy^2), 2\pi \cos(\pi xy^2)) \) which is at the point \((1, 1)\) equal to \( \nabla f(1, 1) = (\pi \cos(\pi), 2\pi \cos(\pi)) = (-\pi, 2\pi) \).

2. Linearization can be used to estimate functions near a point. In the previous example,

\[
-0.00943 = f(1 + 0.01, 1 + 0.01) \sim L(1 + 0.01, 1 + 0.01) = -\pi \cdot 0.01 - 2\pi \cdot 0.01 + 3\pi = -0.09942 .
\]

3. Here is an example in three dimensions: find the linear approximation to \( f(x, y, z) = x^2 + y^2 + z^2 \) at the point \((1, 1, 1)\). Since \( f(1, 1, 1) = 3 \), and \( \nabla f(x, y, z) = (2x, 2y, 2z) \), we have \( L(x, y, z) = f(1, 1, 1) + (2, 2, 2) \cdot (x-1, y-1, z-1) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3 \).

4. Estimate \( f(0.01, 24.8, 1.02) \) for \( f(x, y, z) = e^{x+y}z \).

Solution: take \((x_0, y_0, z_0) = (0, 25, 1)\), where \( f(x_0, y_0, z_0) = 5 \). The gradient is \( \nabla f(x, y, z) = (e^z, e^x, e^x)(2, 2, 1) \). At the point \((x_0, y_0, z_0) = (0, 25, 1)\) the gradient is the vector \((5, 1, 10, 5)\). the linear approximation is \( L(x, y, z) = f(x_0, y_0, z_0) + \nabla f(x_0, y_0, z_0)(x-x_0, y-y_0, z-z_0) = 5 + (5, 1, 10, 5)(x-0, y-25, z-1) = 5x + y/10 + 5z - 2.5 \). We can approximate \( f(0.01, 24.8, 1.02) \) by \( 5 + (5, 1, 10, 5)(0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13 \). The actual value is \( f(0.01, 24.8, 1.02) = 5.1306 \) very close to the estimate.

5. Find the tangent line to the graph of the function \( g(x) = x^2 \) at the point \((2, 4)\).

Solution: the level curve \( f(x, y) = y - x^2 = 0 \) is the graph of a function \( g(x) = x^2 \) and the tangent at a point \((2, g(2)) = (2, 4) \) is obtained by computing the gradient \( \langle a, b \rangle = \nabla f(2, 4) = (-g'(2), 1) = (-4, 1) \) and forming \(-4x + y = d\), where \( d = -4 \cdot 2 + 1 \cdot 4 = -4 \). The answer is \(-4x + y = -4\) which is the line \( y = 4x - 4 \) of slope 4.

6. The Barth surface is defined as the level surface \( f = 0 \) of

\[
\begin{align*}
\quad f(x, y, z) &= (3 + 5t)(-1 + x^2 + y^2 + z^2)^2(-2 + t + x^2 + y^2 + z^2)^2 + 8(x^2 - t^4 y^2)(-t^4 x^2 + z^2)(y^2 - t^4 z^2)(x^4 - 2x^2 y^2 + y^4 - 2x^2 z^2 - 2y^2 z^2 + z^4),
\end{align*}
\]

where \( t = (\sqrt{5} + 1)/2 \) is a constant called the golden ratio. If we replace \( t \) with \( 1/t = (\sqrt{5} - 1)/2 \) we see the surface to the middle. For \( t = 1 \), we see the right the surface \( f(x, y, z) = 8 \). Find the tangent plane at the latter surface at the point \((1, 1, 0)\).

Answer: We have \( \nabla f(1, 1, 0) = (64, 64, 0) \). The surface is \( x + y = d \) for some constant \( d \). By plugging in \((1, 1, 0)\) we see that \( x + y = 2 \).
The quartic surface

\[ f(x, y, z) = x^4 - x^3 + y^2 + z^2 = 0 \]

is called the piriform. What is the equation for the tangent plane at the point \( P = (2, 2, 2) \) of this pair shaped surface? We get \( \langle a, b, c \rangle = \langle 20, 4, 4 \rangle \) and so the equation of the plane

\[ 20x + 4y + 4z = 56, \]

where we have obtained the constant to the right by plugging in the point \( (x, y, z) = (2, 2, 2) \).

**Remark:** some books use differentials etc to describe linearizations. This is 19 century notation and terminology and should be avoided by all means. For us, the linearization of a function at a point is a linear function in the same number of variables. 20th century mathematics has invented the notion of differential forms which is a valuable mathematical notion, but it is a concept which becomes only useful in follow-up courses which build on multivariable calculus like Riemannian geometry. The notion of "differentials" comes from a time when calculus was still foggy in some areas. Unfortunately it has survived and appears even in some calculus books.

**Homework**

1. If \( 2x + 3y + 2z = 9 \) is the tangent plane to the graph of \( z = f(x, y) \) at the point \( (1, 1, 2) \). Estimate \( f(1.01, 0.98) \).
2. Estimate \( 1000^{1/5} \) using linear approximation
3. Find \( f(0.01, 0.999) \) for \( f(x, y) = \cos(\pi xy) + \sin(x + \pi y) \).
4. Find the linear approximation \( L(x, y) \) of the function

\[ f(x, y) = \sqrt{10 - x^2 - 5y^2} \]

at \( (2, 1) \) and use it to estimate \( f(1.95, 1.04) \).
5. Sketch a contour map of the function

\[ f(x, y) = x^2 + 9y^2 \]

find the gradient vector \( \nabla f = \langle f_x, f_y \rangle \) of \( f \) at the point \( (1, 1) \). Draw it together with the tangent line \( ax + by = d \) to the curve at \( (1,1) \).