A short introduction to several complex variables

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Caltech, 1996

This text is a short two hour introduction into the theory of several complex variables. These lectures were given in May 1996 at Caltech to the class Ma 108 substituting for someone else. As prerequisite, the topic requires some familiarity with complex analysis in one dimension.

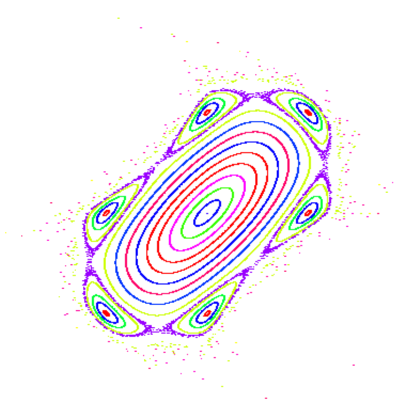
1 Motivation and Plan

The higher dimensional generalization of complex analysis in one variables is an important part of mathematics. Our short visit of this theory will allow us to repeat some facts of the theory in one complex variable.

A **motivating example**, where multi-dimensional complex analysis can occur in mathematical research, is the iteration of **multi-dimensional analytic maps** like for example the **Hénon map family $\mathbf{C^2} \to \mathbf{C^2}$**

$$T : \begin{bmatrix} z \\ w \end{bmatrix} \to \begin{bmatrix} z^2 + c - aw \\ z \end{bmatrix} ,$$

where a, c are complex parameters. If $a = 0$, one obtains, by restriction to the first coordinate, the iteration of a one dimensional map $z \mapsto z^2 + c$.



A theorem in dynamical system theory assures that the fixed point (z_c, w_c) near the center is stable: if we start iterating with an initial condition near that point, we stay near that point. This stability does no more hold in $\mathbf{C^2}$. The theory to understand the iteration of the Hénon map in $\mathbf{C^2}$ heavily needs the theory of complex variables in several dimensions.

The **aim** of this two hour introduction is

1. to show that part of complex analysis in several variables can be obtained from the one-dimensional theory essentially by replacing indices with multi-indices. Examples of results which extend are Cauchy's theorem, the Taylor expansion, the open mapping theorem or the maximum theorem.

2. to show in two examples that there are new features in several dimensions. For example, a multi-dimensional Riemann mapping theorem or a multi-dimensional Piccard theorem does no

more hold. The theory becomes richer in higher dimensions.

2 Holomorphic functions

Notation. $\mathbf{C^n}$ denotes the n-dimensional complex vector space. A point in $\mathbf{C^n}$ is written as $z = (z_1, z_2, \ldots, z_{n-1}, z_n)$, where $z_j = x_j + iy_j$ are complex numbers. We write

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})f$$

for the complex partial derivatives of a function $f : \mathbf{C^n} \to \mathbf{C}$.
We also use the notation for $n \in \mathbf{N^n}, \mathbf{z} \in \mathbf{C^n}$

$$\frac{\partial^m f}{(\partial z)^m} = \frac{\partial^{m_1}}{(\partial z_1)^{m_1}} \cdots \frac{\partial^{m_1}}{(\partial z_n)^{m_n}} f(z) \ .$$

Definition. Let $D \subset \mathbf{C^n}$ be an open set. A continuous function $f : D \to \mathbf{C}$ is called **holomorphic in D** or **analytic in D**, if for all $z \in D$ and $1 \leq j \leq n$, the complex partial derivatives $\frac{\partial f}{\partial z_j}$ exist and are finite.

Remark. In other words, a function $D \to \mathbf{C}$ is holomorphic if it is continuous and holomorphic in each of its variables.

Proposition 2.1 (Basic properties of holomorphic maps)

If f, g are holomorphic in D, then $f + g, f - g, f \cdot g$ are holomorphic in D. If $g(z) \neq 0$ for all $z \in D$, then also f/g is holomorphic in D.
If f_n is a sequence of holomorphic maps in D which converges uniformly on compact subsets of D to f, then f is holomorphic in D.

Proof. These facts follow directly from the one-dimensional theory: if f is holomorphic in D, then it is holomorphic in each of the variables separately. That is, if we fix $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$, then the map $z_j \mapsto f(z_1, \ldots, z_j, \ldots, z_n)$ is holomorphic as a map from $D_j \to \mathbf{C}$, where $D_j = \{z_j \mid (z_1, \ldots, z_j, \ldots, z_n) \in D\}$. ■

Examples.
1) The function $f : (z_1, z_2) \mapsto 2z_1^3 + \sin(z_1) - z_2^5$ is holomorphic on $D = \mathbf{C^2}$.

2) The function $f : (z_1, z_2) \mapsto e^{z_1^2 + z_2^2}$ is holomorphic in $\mathbf{C^2}$. We will see in a moment that it can be written as $f(z_1, z_2) = \sum_{k_1,k_2=0}^{\infty} \frac{z_1^{k_1} z_2^{k_2}}{k_1! k_2!}$.

3) The function $f : (z_1, z_2) \mapsto \sum_{k=0}^{\infty} z_2 z_1^k$ is holomorphic in $D = \{|z_1| < 1\} \times \mathbf{C}$.

4) The function $f : (z_1, z_2) \mapsto \frac{z_2}{1 - z_1}$ is holomorphic in $\mathbf{C} \setminus \{\mathbf{1}\} \times \mathbf{C}$. It is the **analytic continuation** of 3) from D to $\mathbf{C} \setminus \{\mathbf{1}\} \times \mathbf{C}$.

Short exercise. Find the region in $\mathbf{C^2}$, where the sum

$$f(z_1, z_2) = \sum_{k=0}^{\infty} z_1^k z_2^k$$

is analytic. Find an explicit analytic continuation of f to a larger domain.

Remark. It is actually not necessary to assume that f is continuous: a theorem of Hartogs assures that the continuity assumption is redundant in the definition of analyticity. Note the difference between the real and the complex multi-dimensional case, because as you well know, a real function $f : \mathbf{R^n} \to \mathbf{R}$ for which all partial derivatives exist, needs not to be continuous:

Definition. Let $K \subset \mathbf{C^n}$ be any set. A function $f : K \to \mathbf{C}$ is called **holomorphic** on K if each point $a \in K$ has an open neighborhood D such that $D \cap K$ is closed and such that there exists a function f_D which is holomorphic on D such that $f_D = f$ on $D \cap K$.

Example. If f is holomorphic on D and K is a closed subset of D then f is holomorphic on K. Proof. Take for any point $a \in K$ the open set D.

3 Cauchy, Taylor and Co.

Definition. A **closed polydisc** $\overline{\mathbf{D}(\mathbf{a},\mathbf{r})} = \overline{\mathbf{D^n}(\mathbf{a},\mathbf{r})}$ with center $a \in \mathbf{C^n}$ and radius r is the set $\{z \in \mathbf{C^n} \mid |\mathbf{z_j} - \mathbf{a_j}| \leq \mathbf{r_j}, \mathbf{j} = \mathbf{1}, \dots, \mathbf{n}\ \}$. The interior $\mathbf{D}(\mathbf{a},\mathbf{r})$ is called the **open polydisc** or simply **polydisc**.

The **boundary** of the polydisc is the topological boundary. It consists of the set of points in the closed polydisc which satisfy $|z - a_j| = r_j$ for some r.

The **distinguished boundary** $\mathbf{T^n}(\mathbf{a},\mathbf{r})$ of the polydisc is the set $\{z | z_j = a + re^{i\theta_j}, 0 \leq \theta_j < 2\pi\}$.

If $a = 0, r_j = 1$, we use also the notation $\mathbf{D^n}$ or $\mathbf{T^n}$.

We denote by $\mathbf{B^n}(\mathbf{a},\mathbf{r}) = \{\mathbf{z} \in \mathbf{C^n} \mid \sum_{\mathbf{j=1}}^{\mathbf{n}} |\mathbf{z_j}|^{\mathbf{2}} = \mathbf{r^2}\ \}$ the ball with radius $r \in \mathbf{R}^+$ around a.

Figure 3a) The polydisc $\mathbf{D} \subset \mathbf{C^2}$ shown in the the coordinates $|z_1|, |z_2|$.

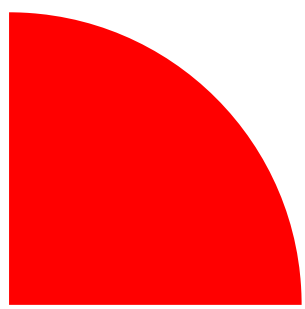


Figure 3b) The ball $\mathbf{B} \subset \mathbf{C^2}$ shown in the the coordinates $|z_1|, |z_2|$.

Remark. Note that the distinguished boundary of the polydisc is topologically a real n-dimensional torus $\mathbf{T^n}$, whereas the boundary has real dimension $2n - 1$.

Theorem 3.1 (Cauchy Formula)

Let f be holomorphic on the closed polydisc $\overline{\mathbf{D}}^n(a,r)$. Then, for all $z \in \mathbf{D^n}(\mathbf{a},\mathbf{r})$ one has

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\mathbf{T^n}(\mathbf{a},\mathbf{r})} \frac{f(w)}{w - z}\, dw\ .$$

We used the notation $dw = dw_1 dw_2 \dots dw_n$ and $\frac{1}{w-z} = \prod_{j=1}^{n} \frac{1}{w_j - z_j}$.

Proof. The proof goes by induction on the dimensions. For $n = 1$, it is the usual Cauchy formula. Assume the formula has been proven in dimension $n - 1$. Define the analytic function $g(z) = f(z, z_2, \dots, z_n)$. By the one dimensional Cauchy formula applied to the function g on the disc $\{|z_1 - a_1| < r_1\}$, one has

$$f(z_1, z_2, \dots, z_n) = \frac{1}{(2\pi i)} \int_{\mathbf{T(a_1,r_1)}} \frac{f(w_1, z_2, \dots, z_n)}{w_1 - z_1} \, dw_1 \; . \tag{1}$$

If we fix w_1, the function $(z_2, \dots, z_n) \to f(w_1, z_2, \dots, z_n)$ is a function of $n - 1$ variables. By the induction assumption, we have

$$f(w_1, z_2, \dots, z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{\mathbf{T^{n-1}(a,r)}} \frac{f(w_1, z_2, \dots, z_n)}{w - z} \, dw_2 \dots dw_n \; . \tag{2}$$

If we plug in Equation 2 into the right hand side of Equation 1, we obtain the Cauchy's formula in n dimensions. ∎

Corollary 3.2 (Holomorphic maps are smooth)

If f is holomorphic, then $(\partial/\partial z_j) f$ is holomorphic. Moreover, all partial derivatives $\partial^n f/(\partial z)^n$ exist.

Proof. Differentiate on the right hand side of the Cauchy formula gives

$$f^{(k)}(z) = \frac{1}{(2\pi i)^n} \int_{\mathbf{T^n(a,r)}} \frac{f(w)}{(w - z)^{k+1}} k! \, dw \; . \tag{3}$$

∎

<u>More notation</u>. We use the notation $k = (k_1, k_2, \dots, k_n)$ and say $k \geq 0$ if all coordinates of the multi-index k satisfy $k_j \geq 0$. We also use the abbreviation $(z - a)^k = \prod_{j=1}^{k} (z_j - a_j)^{k_j}$ and

$$f^{(k)}(a) = \frac{\partial^k f}{(\partial z)^k}(a)$$

as well as $k! = k_1! k_2! \cdots k_n!$.

Corollary 3.3 (Cauchy estimates)

Assume f is holomorphic in $\overline{\mathbf{D(a,r)}}$ and that $|f(z)| \leq M$ for all $z \in \overline{\mathbf{D(a,r)}}$. Then, for $a \in \mathbf{D(a,r)}$, we have

$$|f^{(k)}(a)| \leq \frac{M k!}{r^k} \; .$$

Proof. This follows directly from the Cauchy formula by majorizing the integral: replace f inside the integral by M and do the estimate! ∎

Corollary 3.4 (Taylor Series Expansion)

If f is holomorphic in the open polydisc $\mathbf{D(a,r)}$, then

$$f(z) = \sum_{k \geq 0} \frac{1}{k!} f^{(k)}(a)(z - a)^k \; .$$

Proof. The proof is the same as in one variable: expand $z \to \frac{1}{w-z}$ into a power series

$$\frac{1}{w-z} = \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}} .$$

(Remind the notation $z^k = \prod_{j=1}^n z_j^{k_j}$ also in the following).
By Cauchy's formula and the Cauchy formula for the derivative, one has

$$\begin{aligned} \sum_k \frac{f^{(k)}(a)}{k!}(z-a)^k &= \frac{1}{(2\pi i)^n} \int_{\mathbf{T^n(a,r)}} f(w) \sum_k \frac{(z-a)^k}{(w-a)^{k+1}} \, dw \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbf{T^n(a,r)}} \frac{f(w)}{w-z} \, dw = f(z) . \end{aligned}$$

■

Corollary 3.5 (Principle of analytic continuation)

Assume f is holomorphic in an open connected subset (=domain) of $\mathbf{C^n}$. If f vanishes on an open subset of D, then $f = 0$ on D.

Proof. The same proof as in one variable shows that the set $E = \{z \in D \mid f^{(n)}(z) = 0, \forall n \in \mathbf{N^n}\}$ is closed as a countable intersection of closed sets $E_n = \{z \in D \mid f^{(n)}(z) = 0\}$. On the other hand, if $z \in E$, and U is a neighborhood of z in which the Taylor series expansion holds, then $f(z) = 0$ in U so that E is also open. As a both open and closed set, $E = D$ or $E = \emptyset$. ■

Corollary 3.6 (Open mapping theorem)

If D is a domain in $\mathbf{C}$ and $f : D \to \mathbf{C}$ is not constant and holomorphic on D, then f is open (=it maps open sets into open sets).

Proof. Given $a \in D$ and let $\mathbf{B(a,r)} \subset \mathbf{D}$. By the principle of analytic continuation, f is not constant on U (because it were then also on D). There exists $b \in \mathbf{B(a,r)}$ with $f(b) \neq f(a)$. Define the function in one variable $g(z) = f(a + z(b-a))$ for $z \in U = \{z \in \mathbf{C} \mid \mathbf{a} + \mathbf{z}(\mathbf{b} - \mathbf{a}) \in \mathbf{D}\}$. This function g is not constant and by the open mapping theorem in one dimensions, we know that $g(U)$ is open in $\mathbf{C}$. Since $g(U) \subset g(D)$, the result follows. ■

Corollary 3.7 (Maximum principle)

Let D be a domain in $\mathbf{C^n}$ and let f be holomorphic on $\overline{D}$. Then $|f(z)|$ takes its maximum on the boundary δD of D.

Proof. Assume f is not constant on D. Define $M = \sup_{z \in \delta D} |f(z)|$. By the open mapping theorem and the boundedness of f, we know that $f(D)$ is a bounded open domain in $\mathbf{C}$ and that f maps the boundary of D into the boundary of $f(D)$. ■

The proof of the following theorem is left as an exercise.

Corollary 3.8 (Montel's theorem)

Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a family of holomorphic functions on a domain $D \subset \mathbf{C^n}$. Assume, for every compact $K \subset D$, there exists a constant M_K such that $|f(z)| \leq M_K$ for all $z \in K$ and $f \in \mathcal{F}$. Then, for every sequence f_n in $\mathcal{F}$, there is a subsequence, which converges uniformly on compact subsets.

Proof. Use Cauchy estimates to give bounds on the Taylor coefficients. ■

In the next hour, we will consider two situations, where the theory of complex variables differs from the one dimensional theory (and gets so more interesting!).

4 A theorem of Poincaré

Definition Given $D \subset \mathbf{C^n}$. A map $f = (f_1, \dots, f_m) : D \to \mathbf{C^m}$ is called **holomorphic**, if each of the coordinate functions $f_k : D \to \mathbf{C}$ is holomorphic. A map $f : D_1 \to D_2$ is called **biholomorphic**, if it is holomorphic and if there is an inverse which is holomorphic. If there exists a biholomorphic map $f : D_1 \to D_2$, then D_1 is called **biholomorphic to** D_2.
The group of biholomorphic maps from a domain onto itself is called $\mathrm{Aut}(D)$. (Convince yourself that $\mathrm{Aut}(D)$ is indeed a group!) Given $a \in D$, one can form the subgroups $\mathrm{Aut}_a(D)$ of biholomorphic maps on D which leave a invariant.

Remember that the **Rieman mapping theorem** in one complex variable assures that every smoothly bounded, simply connected open domain D in $\mathbf{C}$ is biholomorphic to the unit disc.

There is no analogue of such a result in higher dimensions. Our aim is to prove the following theorem of Poincaré which shows that the analysis of holomorphic functions in several complex variables depends in general on the domain in question.

Proposition 4.1 (Poincaré)

The ball $\mathbf{B^n}$ is not biholomorphic to the polydisc $\mathbf{D^n}$ for $n \geq 2$.

Definition. For bounded D, the group $Aut(D)$ becomes a topological group by defining a distance between two automorphisms $d(\sigma_1, \sigma_2) = \sup_{z \in D} |\sigma_1(z) - \sigma_2(z)|$. Denote by $Aut^{Id}(D)$ the group of automorphism which are connected to the identity.

The proof will need some preparation. The basic idea of Poincaré was to look at the groups $\mathrm{Aut}(D)$ of biholomorphic maps of D and to realize:

Lemma 4.2 *If D_1 is biholomorphic to D_2, then the groups $\mathrm{Aut}(D_1)$ and $\mathrm{Aut}(D_2)$ are isomorphic groups. Given $a_1 \in D_1$ and $a_2 \in D_2$ for which there exists a biholomorphic map $f : D_1 \to D_2$ with $f(a_1) = f(a_2)$, then $\mathrm{Aut}_{a_1}(D_1)$ and $\mathrm{Aut}_{a_2}(D_2)$ are isomorphic groups. Also $Aut^{Id}(D_1)$ and $Aut^{Id}(D_2)$ are isomorphic as well as $Aut_a^{Id}(D_1)$ and $Aut_a^{Id}(D_2)$.*

Proof. Let $f : D_1 \to D_2$ be a biholomorphic map from D_1 to D_2, then

$$\sigma \mapsto f \circ \sigma \circ f^{-1}$$

is a group homomorphism from $\mathrm{Aut}(D_2)$ to $\mathrm{Aut}(D_1)$. Because it is invertible, it is a group isomorphism. ∎

Definition. Denote with $SU(n)$ the group of all $n \times n$ matrices A which are unitary $AA^* = 1$ and which have determinant 1. It is called the **special unitary group**.

Proposition 4.3

$\mathrm{Aut}_0^{Id}(\mathbf{B^n})$ *is nonabelian.*

Proof. The nonabelian special unitary group $SU(n)$ is a subgroup of $\mathrm{Aut}_0^{Id}(\mathbf{B^n})$, because a matrix $A \in SU(n)$ defines the biholomorphic map $z \mapsto Az$ on $\mathbf{B^n}$ which leaves 0 invariant. ∎

Proposition 4.4

For every $a \in \mathbf{D^n}$, the group $\mathrm{Aut}_a^{Id}(\mathbf{D^n})$ is abelian.

Postponing the proof of Proposition 4.4, we can understand Poincaré's result:

Proof. Assume $D_1 = B^n(0,1)$ is biholomorphic to $D_2 = \mathbf{D^n(0,1)}$. From Lemma 4.2 and the transitivity of $\mathrm{Aut}(D_2)$, we conclude that $\mathrm{Aut}_0^{Id}(D_1)$ and $\mathrm{Aut}_{f(0)}^{Id}(D_2)$ are isomorphic groups. But Proposition 4.3 says that $\mathrm{Aut}_0^{Id}(D_1)$ is nonabelian while Proposition 4.4 states that $\mathrm{Aut}_{f(0)}^{Id}(D_2)$ is abelian. This is a contradiction. ■

In order to prove of Proposition 4.4 we need to know more about biholomorphic maps of bounded domains.

Definition. If k is a multi-index, define $|k| = \sum_{j=1}^n k_j$. A function $f(z) = \sum_{|k|=N} a_k z^k$ is called a **homogeneous polynomial of degree** N.

Proposition 4.5 (Cartan uniqueness theorem)

Let D be a bounded domain in $\mathbf{C^n}$ and given $a \in D$. If $f \in \mathrm{Aut}_a(D)$ satisfies $f'(a) = 1$, then $f(z) = z$ for all $z \in D$.

Proof. By a translation of coordinates (replace D with $D - a$), we can assume $a = 0$. Because D is bounded, one has $\overline{D} \subset \mathbf{D^n(0,R)}$ for some $R > 0$. Every $f \in \mathrm{Aut}_0(D)$ has a Taylor expansion at the origin $f(z) = \sum_n a_n z^n$. Cauchy's estimates give $|a_n| \leq M r^{-n}$, where r is such that $\mathbf{D^n(0,r) \subset D}$ and $M = \sup_{z \in \overline{D}} |f(z)|$. By assumption, f has a Taylor expansion

$$f(z) = z + f_N(z) + \dots$$

where f_k are n tuples of homogeneous polynomials of degree k and N is chosen to be the smallest possible. The k'th iterate $f^k = f \circ \dots \circ f$ of f has then the Taylor expansion

$$f^k(z) = z + k \cdot f_N(z)$$

which violates the above Cauchy estimate for large k unless $f_N = 0$. But if $f(z) = z$ in $D(0,r)$, then also $f(z) = z$ in D by the principle of analytic continuation. ■

Definition A bounded domain $D \subset \mathbf{C^n}$ is called a **circular domain**, if $z \in D$ implies that $k_\theta(z) = e^{i\theta}$ for all $z \in D$ and all $\theta \in \mathbf{R}$.

Corollary 4.6 (Cartan)

Let D be a bounded circular domain in $\mathbf{C^n}$ and assume $0 \in D$ and $f \in \mathrm{Aut}_0(D)$. Then f is linear.

Proof. Because D is a circular domain and $0 \in D$, we have $k_\theta \in \mathrm{Aut}_0(D)$. Define

$$g = k_{-\theta} \circ f^{-1} \circ k_\theta \circ f \ .$$

We have $g'(0) = k'_{-\theta}(0) \circ f'^{-1}(0) \circ k'_\theta \circ f'(0) = Id$ so that by the previous proposition $g(z) = z$. This implies

$$k_\theta \circ f = f \circ k_\theta \ .$$

If $f = (f_1, \dots, f_n)$, then $f_j(e^{i\theta} z) = e^{i\theta} f_j(z)$. Let $f_j(z) = \sum_k a_k z^k$. Then

$$e^{i\theta} a_k = e^{i|k|\theta} a_k \ .$$

This implies $a_k = 0$ for all $|k| \geq 1$. ■

Proposition 4.4 follows immediately from

Corollary 4.7

Every $f = (f_1, \dots, f_n) \in \mathrm{Aut}(\mathbf{D^n})$ has the form

$$f_j(z) = e^{i\theta_j} \frac{z_{p(j)} - a_j}{1 - \overline{a}_j z_{p(j)}} , \tag{4}$$

where $\theta_j \in \mathbf{R}, \mathbf{a} \in \mathbf{D^n}$ and where p is a permutation of the multi-index $j = (j_1, ..., j_n)$.

Proof. The map defined in Equation (4) is clearly an automorphism. Denote such an automorphism by σ_a, if $\theta_j = 0$ and $p = \mathrm{Id}$. Given $f \in \mathrm{Aut}(\mathbf{D^n})$, the automorphism $\sigma_a \circ f$ leaves 0 invariant. We can therefore assume that $f \in \mathrm{Aut}_0(\mathbf{D^n})$. Because $\mathbf{D^n}$ is a circular domain, the previous corollary assures that f is linear: $f_k(z) = \sum A_{kj} z_j$. Because $f(\mathbf{D^n}) \subset \mathbf{D^n}$, we have $\sum_{k=1}^n |A_{kj}| \leq 1$. However, by choosing sequences $z^{(n)} = (0, \dots, 0, 1 - \frac{1}{n}, 0, \dots, 0)$ converging to the distinguished boundary $\mathbf{T^n}$ of $\mathbf{D^n}$, also $f(z^{(n)}) = (1 - \frac{1}{n})(A_{1j}, \dots, A_{nj})$ converges to the distinguished boundary of $\mathbf{D^n}$ and so $|A_{q(j)j}| := \max_{k=1,\dots,n} |A_{kj}| = 1$. Because $\sum_{k=1}^n |A_{kj}| \leq 1$, we know that A_{jk} is a permutation matrix which has non-vanishing entries of norm 1 only at places $A_{q(j)j}$. If p is the inverse permutation of q, then $f_k(z) = A_{k,p(k)} z_{p(k)}$ with $|A_{k,p(k)}| = 1$. ∎

5 Fatou-Biberbach domains

Definition A subset $U \subset \mathbf{C^2}$ such that U is biholomorphic to $\mathbf{C^2}$ and such that $\mathbf{C^2} \setminus \mathbf{U}$ contains an open subset in $\mathbf{C^2}$ is called a **Fatou-Biberbach domain**.

Picard theorem in one complex variable states that a map which is analytic in $\mathbf{C}$ and which omits two values must be constant. Consequently, there exists no proper subset $U \subset \mathbf{C}$ which is biholomorphic to $\mathbf{C}$ and such that $\mathbf{C} \setminus \mathbf{U}$ contains an open subset. For a proof, see see Alfors, p. 307. In other words, there exist no analogues of Fatou-Biberbach domains in $\mathbf{C}$.

Proposition 5.1

There exist Fatou-Biberbach domains $D \subset \mathbf{C^2}$.

Proof. (i) Set-up of a class of dynamical systems.
Let $p(z)$ be a polynomial which satisfies $p(0) = p'(0) = 0$ like for example $p(z) = z^3$. Let $\lambda > 1$ be a real number. Consider the analytic map

$$S : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} z_2 \\ -\lambda^2 z_1 - p(z_2) \end{pmatrix} .$$

This map is invertible and has the inverse

$$S^{-1} : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} -\lambda^{-2}(z_2 + p(z_2)) \\ z_1 \end{pmatrix} .$$

(Note that for $p(z) = \lambda^2 z^2 + c\lambda^2$, $a = \lambda^{-2}$, this is the Hénon map mentioned at the beginning of these notes).

(ii) Definition of a domain D.
The Jacobean $DS(0,0)$ is the linear map

$$\Lambda := DS(0,0) = \begin{pmatrix} 0 & 1 \\ -\lambda^2 & 0 \end{pmatrix} .$$

It has the eigenvalues $\pm\lambda$. We see that S^{-1} is a contraction near 0. The **basin of attraction** is defined as the set

$$D = \{(z_1, z_2) \mid S^{-j}(z_1, z_2) \to 0\} .$$

(iii) D is biholomorphic to $\mathbf{C^2}$.
There exists an analytic function u defined in a neighborhood of $(0,0)$, which conjugates there S to its linear part:

$$S \circ u = u \circ \Lambda \ .$$

The construction of u is a nice exercise. u is explicitly given as the limit of functions $u(z) = \lim_{l\to\infty} S^{-l} \circ \Lambda^l(z)$. One has to show that this limit exists and that it indeed satisfies $S \circ u = u \circ \Lambda$. By the formula

$$u(z_1, z_2) = S^m \circ u \circ \Lambda^{-m}(z_1, z_2)$$

we can define u on the whole $\mathbf{C^2}$ and D is the image of u. Since u is invertible near $(0,0)$, we can also invert u on D.

(iv) Find p such that D omits an open subset of $\mathbf{C^2}$.
If the polynomial p is such that $p(1) = \lambda^2 - 1, p'(1) = 0$, then $(1,1)$ is a fixed point and

$$DS(1,1) = \begin{pmatrix} 0 & 1 \\ -\lambda^2 & 0 \end{pmatrix} \ .$$

This shows that S^{-1} is also a contraction near $(1,1)$ and a neighborhood of $(1,1)$ is attracted to $(1,1)$ and not to $(0,0)$.

[Alternatively for (iv), one could also consider domains $E = \{(z_1, z_2) \mid |z_1| > R, |z_2| < |z_1| \ \}$. If (z_1, z_2) is in E, then the norm of $S^n(z_1, z_2)$ diverges.] ∎

6 Domains of holomorphy

Definition An open set U in $\mathbf{C^n}$ is a **domain of holomorphy**, if one can not find two nonempty open sets $U_1 \subset U_2$ such that U_2 is connected and not contained in U, $U_1 \subset U_2 \cap U$ and so that for every holomorphic function h on U, there is a holomorphic function h_2 on U_2, which coincides with h on U_1.

Proposition 6.1

In one complex variable, every open set is a domain of holomorphy,

Proof.
(i) The unit disc $\mathbf{D} = \{|\mathbf{z}| \leq \mathbf{1}\}$ is a domain of holomorphy.
The function

$$f(z) = \sum_n 2^{-j} z^{2^j}$$

is analytic in $\mathbf{D}$ and continuous in $\overline{\mathbf{D}}$. However, on the boundary function on $\mathbf{T}$ of $\overline{\mathbf{D}}$

$$\theta \mapsto \sum_{j=0}^{\infty} 2^{-j} e^{i 2^j \theta}$$

is nowhere differentiable. One can therefore not extend f to a larger domain. This proves that $\mathbf{D}$ is a domain of holomorphy.

(ii) Every simply connected smoothly bounded domain D is a domain of holomorphy.
By the Riemann mapping theorem, there are analytic maps $\phi : D \to \mathbf{D}, \phi^{\mathbf{-1}} : \mathbf{D} \to \mathbf{D}$, which are continuously differentiable on the closure of the domains. The function $f \circ \phi : D \to \mathbf{C}$ is now an analytic map on D which can not be extended to a larger domain.

(iii) Every open set is a domain of holomorphy.
Take a sequence of points z_j which have no accumulation point in D but such that every boundary poet of D is an accumulation points. There is an analytic function which is vanishing on each of the points z_j and nowhere else. This function has no analytic continuation on a strictly larger set, because such a function would have a zero set with accumulation points in the interior of D and f would therefore vanish. ■

In higher dimensions, this is different:

Proposition 6.2 (Example of Hartogs extension phenomenon)

Every holomorphic function f on $D = \mathbf{D^2(0,3)} \setminus \mathbf{D^2(0,1)} \subset \mathbf{C^2}$ extends to a holomorphic function on $\mathbf{D^2(0,3)}$. Especially, D is not a domain of holomorphy.

Proof. Let f be holomorphic on D. For fixed z_1, we can write $f(z_1, z_2)$ in a Laurent expansion

$$f(z_1, z_2) = \sum_{j=-\infty}^{\infty} a_j(z_1) z_2^j$$

with

$$a_j(z_1) = \frac{1}{2\pi i} \int_{|z_2|=2} \frac{f(z_1, w)}{w^{j+1}} \, dw$$

which depends analytically on z_1. Because f is analytic in D, we have

$$a_j(z_1) = 0, j < 0, 1 < |z_1| < 3$$

We make an analytic continuation of a_j by defining $a_j(z_1) = 0$ for $j < 0$ and all z_1. The function

$$h(z_1, z_2) = \sum_{j=0}^{\infty} a_j(z_1) z_2^j$$

defines an analytic function on $\mathbf{D^2(0,3)}$ which coincides with f on D. ■

A basic problem is to **characterize domains of holomorphy**. Part of the problem is **Levi's problem**, the characterization of such domains with respect to geometric properties of the boundary.

Problems

1) a) Find the region in $\mathbf{C^2}$, where the sum

$$f(z_1, z_2) = \sum_{k=0}^{\infty} z_1^k z_2^k$$

is analytic.
b) Find an explicit analytic continuation of f to a larger domain and show that g is unique.
c) Find the region in $\mathbf{C^2}$, where the sum

$$g(z) = \sum_{k=0}^{\infty} z^k$$

is analytic. (Notice that $k = (k_1, k_2)$ is a multi-index.)
d) Find an explicit analytic continuation of g to a larger domain and show that g is unique.

Purpose: work with multi-dimensional Taylor series and multi indices. Also a reminder what analytic continuation means in a simple situation.

More hints:
a) $|z_1 z_2| < 1$.
b) $1/(1 - z_1 z_2)$ defined on $D = \mathbf{C^2} \setminus \{\mathbf{z_1 z_2 = 1}\}$. Because D is connected, one has uniqueness by the principle of analytic continuation.
c) $\mathbf{D^2}$.
d) $(1 - z)^{-1} = (1 - z_1)^{-1}(1 - z_2)^{-1}$ defined on $D = \mathbf{C^2} \setminus \{\mathbf{z_1 = 1, z_2 = 1}\}$. Because D is connected, one has uniqueness by the principle of analytic continuation.

2) Prove Montel's theorem:
Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a family of holomorphic functions on a domain $D \subset \mathbf{C^n}$. Assume, for every compact $K \subset D$, there exists a constant M_K such that $|f(z)| \leq M_K$ for all $z \in K$ and $f \in \mathcal{F}$. Then, for every sequence f_n in $\mathcal{F}$, there is a subsequence, which converges uniformly on compact subsets.
Hint: Use Cauchy estimates to give bounds on the Taylor coefficients.

Purpose: many one-dimensional theorems immediately generalize to higher dimensions. If the proof of the one dimensional theorem is known, a hint would not be necessary.

More hints: see Narasimhan p.8)

3) Show that for all $a, b \in \mathbf{C^n}$, $r \in (\mathbf{R^+})^\mathbf{n}, \rho \in \mathbf{R^+}$, the domains $\mathbf{D(a, r)}$ and $\mathbf{B(b}, \rho)$ are not biholomorphic.

Purpose: understand Poincaré's result and the definition of biholomorphisms.

More hints: show that all polydiscs are biholomorphic and that all balls are biholomorphic. Use Poincaré's result.