Lecture 6: Worksheets

We stack disks onto each other building \( n \) layers and count the number of discs. The number sequence we get are called \textbf{triangular numbers}.

\[
1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 36 \ 45 \ ...
\]

This sequence defines a \textbf{function} on the natural numbers. For example, \( f(4) = 10 \).

1 Can you find \( f(200) \)? The task to find this number was given to Carl Friedrich Gauss in elementary school. The 7 year old came up quickly with an answer. How?

\textbf{Tetrahedral numbers}

We stack spheres onto each other building \( n \) layers and count the number of spheres. The number sequence we get are called \textbf{tetrahedral numbers}.

\[
1 \ 4 \ 10 \ 20 \ 35 \ 56 \ 84 \ 120 \ ...
\]

Also this sequence defines a \textbf{function}. For example, \( g(3) = 10 \). But what is \( g(100) \)? Can we find a formula for \( g(n) \)?
2. Verify that $g(n) = n(n + 1)(n + 2)/6$, satisfies $Dg(n) = g(n) - g(n - 1) = n(n + 1)/2$.

3. **Problem:** Given the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ which satisfies the rule $f(x) = f(x - 1) + f(x - 2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$, if we assume $f(0) = 0$?

4. **Problem:** Take the same function $f$ given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first $n$ numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, \ldots$. What sequence is that?

5. **Problem:** Find the next term in the sequence

   $2 \ 6 \ 12 \ 20 \ 30 \ 42 \ 56 \ 72 \ 90 \ 110 \ 132$.

6. **Problem:** Find the next term in the sequence

   $3, 12, 33, 72, 135, 228, 357, 528, 747, 1020, 1353, \ldots$.

   To do so, compute successive derivatives $g = Df$ of $f$, then $h = Dg$ until you see a pattern.