Roots

We start with a theorem discovered by Hippasus of Metapontum from 500 BC. Legend tells that the discoverer had to pay with his life for the discovery of incommensurable magnitudes:

\[ \sqrt{2} \text{ is irrational} \]

Assume \( \sqrt{2} = p/q \), then \( q\sqrt{2} = p \) and \( 2q^2 = p^2 \). Since the number of factors 2 on the left are odd and even on the right, this is a contraction.

This works for any \( \sqrt{n} \) as long as \( n \) is not a square. Theodorus of Cyrene, a contemporary of Hippasus who extended some irrationality proofs as we know from his students Theatetus of Cyrene and Plato. The just given irrationality proof relies on the fundamental theorem of arithmetic proven by the 21 year old Karl-Friedrich Gauss in 1798 in his *Disquisitiones Arithmeticae*. In our slides we show a geometric proof by decent which does not need the unique prime factorization result.

Logarithms

Logarithms were introduced independently by the English mathematician John Napier 1550-1617 and the Swiss mathematician Joost Bürgi 1552-1632. Logarithms invert the exponential. For example, \( \log_{10}(1000) = 3 \) because \( 10^3 = 1000 \). An other example is \( \log_2(1/64) = -6 \) as \( 2^{-6} = 1/64 \) or \( \log_{10}(\sqrt{10}) = 1/2 \).

\[ \log_{10}(2) \text{ is irrational} \]

If \( \log(2) = p/q \), then \( 2 = 10^{p/q} \) and so \( 2^q = 10^p \). The right hand side is divisible by 5, the left not.

**Question**: Is \( \log_4(2) \) rational or irrational? **Answer**: it is rational. Can you see why?

Open problem: Is there an irrational \( x \) such that \( 2^x \) and \( 3^x \) are integers?

This would mean that there exist two integers \( n, m \) such that \( \log_2(n) = \log_3(m) \). We could try to search with a computer by hunting down integer pairs \( n, m > 1 \) but it could be a futile hunt because most likely there does not exist such a pair.

Powers
One knows that $\pi$ and the Euler constant $e$ are irrational. We will look at the proof that $\pi$ is irrational in a moment. The number $G = e^\pi = 23.14...$ is called Gelfond’s constant. Hilbert had asked in his 7th problem whether this number is rational. Since $-1 = e^{i\pi}$, we have $G = (-1)^{(1)}$. It follows from a theorem of Gelfond-Schneider that $G$ is irrational as $(-1)$ is rational and $-i$ is algebraic. Also the Gelfond-Schneider constant $\sqrt{2\sqrt{2}}$ is known to be irrational. We will come back to this. The theorem of Gelfond-Schneider is beyond the scope of this course. It is part of transcendental number theory. Alexander Gelfond (1906-1968) was a Russian mathematician who was the chief cryptograph of the Soviet Navy during WW II. Theodor Schneider (1911-1988) was a German mathematician who was a student of Carl Ludwig Siegel. The later was also the doctoral advisor of Jürgen Moser, who was my own undergraduate thesis advisor. Schneider served around 1960 as the director of the Oberwolfach research institute in the “Schwarzwald”.

$i^i$ is irrational

$$i^i = (e^{i\pi/2})^i = e^{-\pi/2} = 1/\sqrt{e\pi} = 1/\sqrt{G} = 0.20788...$$ Assume this is $p/q$, then $1/G = p^2/q^2$ and $G = q^2/p^2$. But that would imply that the Gelfond’s constant $G$ is rational which we know is not true.

So, we have seen that the "eye for an eye" number is irrational. Quite a metaphor. Here are more problems for which the answers are unknown:

Open problem: Is $\pi^e$ is irrational?

Open problem: Is $\pi^{(\pi^{(\pi^n)})}$ an integer?

For an ultrafinitist (which many computer scientists are by nature), a number like $e(e(e(e(e))))$ does not exist yet as we can not realize it yet in a computer. Note the order of the brackets. While we have no idea whether $x^{(x^{(x^{(x^{(x)})})})}$ is an integer if $x = \sqrt{10}$ (it is inaccessible to us), we know that $(((x^x)^x)^x)^x$ is an integer as it is $\sqrt{10}^{100} = 10^{50}$.

Logic

I learned the following result as a student from Gerhard Jäger who is now in the logic and theory group at the University of Bern in Switzerland. The proof is attributed to Dov Jarden.

There exist irrational $x, y$ such that $x^y$ is rational.

There are two possibilities. Either $z = \sqrt{2\sqrt{2}}$ is irrational or not. In the first case, we have found an example where $x = y = \sqrt{2}$. In the second case, take $x = z$ and take $y = \sqrt{2}$. Now $x^y = \sqrt{2}^2 = 2$ is rational and we have an example.

In this case, we know by Gelfond-Schneider’s theorem that the first case happens.

A related result by Ash and Tan proves that
There exists an irrational $x$ such that $x^x$ is rational.

The function $f(x) = x^x$ maps $(1/e, \infty)$ bijectively to $(f(x), \infty)$. The case that $x$ and $f(x)$ are both rational is extremely rare. It only happens if $(x, f(x)$ is an integer pair of the form $(n, n^n)$: assume $x = p/q$ and $f(x) = a/b$ are reduced fractions, then $(p/q)^p = (a/b)^q$ or $p^a b^q = a^b q^p$ this implies that a prime divides $b^q$ if and only if it divides $q^p$. This implies $q = b = 1$. Having shown that the pair $x, x^x$ both being rational only occurs if $x$ is an integer shows that we can solve for example $x^x = 7/5$ for $x$ and know that $x$ must be irrational.

Pi

Babylonian mathematicians knew that $\pi$ is close to $25/8$. Archimedes proved around 300 BC that $223/71 < \pi < 22/7$ and Ptolemy used in 200 AC the approximation value $377/120$ which is off by $0.000074...$ only. Today, 12.1 trillion digits of $\pi$ are known. A trillion digits corresponds to a 1 TB hard drive. When compressed, one could probably still fit the known digits on a 1 TB hard drive. The Swiss mathematician Johann Lambert proved in 1761 that $\pi$ is irrational. We follow a proof of the Canadian mathematician Ivan Niven from 1946 which uses the fundamental theorem of calculus.

$\pi$ is irrational.

Assume $\pi = p/q$. Take a large $n$. Define $f(x) = x^n(p - qx)^n/n!$, a function whose graph we see below for $n = 7$. The function $F = f - f^{(2)} + f^{(4)} - \cdots$ has the property that $F(0), F(\pi)$ are both integers and that $(F' \sin(x) - F \cos(x))^n = f(x) \sin(x)$. Integrating this gives $\int_0^\pi f(x) \sin(x) \, dx = F(\pi) - F(0)$ which is a positive integer as $f(x) \sin(x)$ is positive between 0 and $\pi$. On the other hand, $f(x) \sin(x) < \pi^n p^n/n!$ is arbitrarily small for large $n$. This contraction shows $\pi = p/q$ is absurd.

Remark: there is a physics connection: write $D$ for the derivative. The equation $(1 + D^2)F = F + F'' = f$ is a driven harmonic oscillator where $f$ is a time dependent force. It is solved by $F = (1 - D^2 + D^4 - D^6... )f$, which is a finite sum as $f$ is a polynomial. The rationality of $\pi$ implies that $f(0) = f(\pi) = 0$ and $f^{(2k)}$ are all integers. The fact that $F(\pi) - F(0)$ is a positive integer means that the oscillator’s amplitude has grown by an integer from time $x = 0$ to $x = \pi$. But the rationality of $\pi$ implied the existence of forces $f$ with arbitrary small $\int_0^\pi f(x) \sin(x) \, dx$. For large $n$, there is not enough ”energy” to be pumped into the system to lift the oscillator amplitude by a positive integer.