Lecture 12: Dynamical systems and Chaos

Chaos

Simple transformations can produce chaotic outcomes. Make sure your calculator is in the "Rad" mode. Remember that $2\pi$ radians is equal to 360 degrees. You can check whether your calculator is in Radian mode, by computing $\cos(\pi)$ and get the result $-1$. Make sure your calculator is in rad mode. Use a scientific calculator. In the iphone calculator for example, turn the device to get to the scientific mode.

Order and Chaos with the calculator

1) Take a calculator, and pushing repetitively the button $\cos$. What do you observe?

2) Now repeat pushing the $\sin$ button. What do you observe?

3) Now push $x^2$ repetitively.

4) Now push $\sqrt{x}$ repetitively.

5) What do you see if you push the buttons $\sin$, then type $1/x$ and repeat this process again and again?

6) Experiment with the button $\tan$. Also here, change $\tan$ and $\cot$.

7) Look for other "chaotic" key combinations? Experiment also with Deg and Rad changes and try especially the log functions.
Part II: The Cobweb construction

1. Objective

We graphically compute a few iterates of one dimensional maps.

2. Stability
Lecture 12: The Ulam-Collatz system

1. Objective

We look at a dynamical system of number theoretical nature.

2. The Collatz system

In the Collatz system, we start with an integer and map it with the following rule:

\[ T(x) = \begin{cases} x/2 & \text{if } x \text{ even} \\ 3x + 1 & \text{if } x \text{ odd} \end{cases} \]

The question is whether the orbit always ends up with 1.

For example: \( x = 7 \) produces 7, 22, 11, 34, 17.

3. Experiment

1) Start with the initial condition 26:

2) Start with the initial condition 9:

3) Start with the initial condition 2048:

4) What is wrong with the following proof of the Collatz conjecture?

Proof. Consider only the odd numbers in the Collatz sequence. We show that each odd number is in average 3/4 times smaller than the previous one:

With probability 1/2 the number \( 3x + 1 \) is divisible by 2 and not 4: this increases \( x \) by 3/2
With probability 1/4 the number \( 3x + 1 \) is divisible by 4 and not 8: this decreases \( x \) by 3/4
With probability 1/8 the number \( 3x + 1 \) is divisible by 8 and not 16: this decreases \( x \) by 3/8

To compute the probability, we take logarithms and compute

\[ a = \sum_{n=1}^{\infty} \frac{1}{2^n} \log\left(\frac{3}{2^n}\right) . \]

The average decay rate of the size of a number is the factor \( e^a = 3/4 \).

3) The Collatz system certainly can be modified. Can you find one, for which there is a nontrivial loop?
Lecture 12: Part IV: Cellular automata

1. Objective

We look at a dynamical systems called Cellular automata. These are continuous maps on sequence spaces in which the evolution rule is translational invariant.

2. The Rule 18 CA

<table>
<thead>
<tr>
<th>neighborhood</th>
<th>new middle cell</th>
</tr>
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<tbody>
<tr>
<td>111</td>
<td>0</td>
</tr>
<tr>
<td>110</td>
<td>0</td>
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<tr>
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<tr>
<td>001</td>
<td>1</td>
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<tr>
<td>000</td>
<td>0</td>
</tr>
</tbody>
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Because only cell neighborhoods of the form 100 and 001 lead to an offspring 1, we and 100 = 4, 001 = 1 in binary; we have $2^4 + 2^1 = 18$. 

3. Run it