Lecture 6: Worksheets

We stack disks onto each other building \( n \) layers and count the number of discs. The number sequence we get are called **triangular numbers**.

\[
\begin{align*}
1 & \quad 3 & \quad 6 & \quad 10 & \quad 15 & \quad 21 & \quad 36 & \quad 45 & \quad \ldots
\end{align*}
\]

This sequence defines a **function** on the natural numbers. For example, \( f(4) = 10 \).

Can you find \( f(200) \)? The task to find this number was given to Carl Friedrich Gauss in elementary school. The 7 year old came up quickly with an answer. How?

**Tetrahedral numbers**

We stack spheres onto each other building \( n \) layers and count the number of spheres. The number sequence we get are called **tetrahedral numbers**.

\[
\begin{align*}
1 & \quad 4 & \quad 10 & \quad 20 & \quad 35 & \quad 56 & \quad 84 & \quad 120 & \quad \ldots
\end{align*}
\]

Also this sequence defines a **function**. For example, \( g(3) = 10 \). But what is \( g(100) \)? Can we find a formula for \( g(n) \)?

2. Verify that \( g(n) = n(n+1)(n+2)/6 \), satisfies \( Dg(n) = g(n) - g(n-1) = n(n+1)/2 \).

3. **Problem:** Given the sequence 1, 1, 2, 3, 5, 8, 13, 21, \ldots which satisfies the rule \( f(x) = f(x - 1) + f(x - 2) \). It defines a function on the positive integers. For example, \( f(6) = 8 \). What is the function \( g = Df \), if we assume \( f(0) = 0 \)?

4. **Problem:** Take the same function \( f \) given by the sequence 1, 1, 2, 3, 5, 8, 13, 21, \ldots but now compute the function \( h(n) = Sf(n) \) obtained by summing the first \( n \) numbers up. It gives the sequence 1, 2, 4, 7, 12, 20, 33, \ldots What sequence is that?

5. **Problem:** Find the next term in the sequence

\[
2 \quad 6 \quad 12 \quad 20 \quad 30 \quad 42 \quad 56 \quad 72 \quad 90 \quad 110 \quad 132
\]

6. **Problem:** Find the next term in the sequence

3, 12, 33, 72, 135, 228, 357, 528, 747, 1020, 1353, \ldots

To do so, compute successive derivatives \( g = Df \) of \( f \), then \( h = Dg \) until you see a pattern.