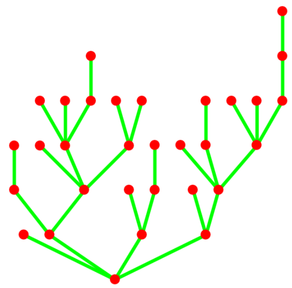
A mathematical theorem

Objective

In this worksheet we want to understand a statement in mathematics, understand the result and see why the result is true.

Trees and Forests

A **graph** is a pair (V, E), where V is a finite set of and where E is a set of vertices, each connecting two different vertices. The set of edges is denoted by E. We assume that edges connect different nodes only once that that no multiple connections appear.
A **closed path** in a graph is a sequence of three or more different vertices, where neighboring vertices are connected by edges and such that at the end we reach the same point again. A graph is called **forest** if it contains no closed path. Part of a graph, where we can go from any vertex to any other is called a **connected component**. A connected component of a forest is called a **tree**.



Trees appear often in applications. Which of the following are trees or can be trees?

- A genealogy map
- A hierarchy in an organization
- A directory structure in a computer
- A protein
- A water molecule
- A DNA string
- Relations between friends
- A computer network
- The internet
- The freeway network

Rules

We assign now numbers called **curvatures** to every node of the tree. We use the following rule:

- At every vertex with only one neighbor (leafs or trunc) put 1/2.
- At a limb, where two branches come together we put 0.
- At a crotch, where $d = 3$ or more branches meet put 1-d/2.

When summing all these curvatures we call this the **total curvature** of the tree. We can summarize this rule as follows:

The curvature of a vertex v is $K(v) = 1 - d(v)/2$ where $d(v)$ is the number of neighbors of v.

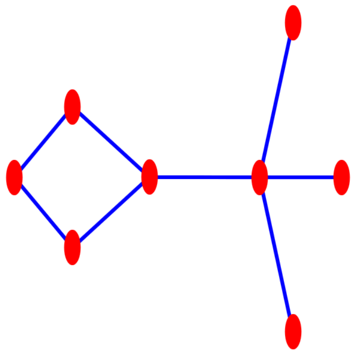
A theorem about trees

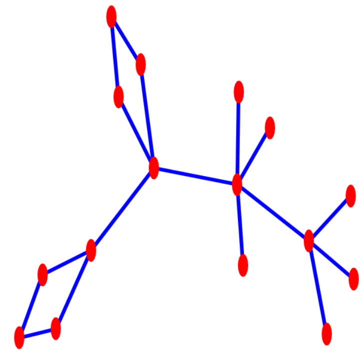
Theorem:
For any forest G, the total curvature is equal to the number of trees.

- Experiment with different trees and forests.
- Start with very simple cases, like graph with 2 or 3 vertices.
- Attempt a proof.

A theorem about gardens

A graph in which no triangles exist is called a **garden**. A connected component of a garden is called a **plant**. Unlike for trees, we can now have closed loops of length larger than 3, which we call **flowers**.





Theorem:
For any garden, the total curvature is the number of plants minus the number of flowers.

- Experiment with different gardens and plants.
- Start with very simple cases, like a plant which has only one flower and no stem.
- Add a stem and see what happens.
- Can you find a proof?