Lecture 1: mathematics

A. About the course

- We have 13 lectures and cover 12 different mathematical fields.
- Historical developments play an important role in this course.
- Difficulties for pioneers developing a field reverberates today in the classroom.
- Quizzes: not competitive, use only classroom resources. Also helps to access
- End project: the top 10 mathematical revolutions?

B. What is Mathematics?

- The 7 liberal arts and their modern interpretation.
- Mathematics is the science of structure.
- Mathematics can be extremely applied.
- Mathematics has changed at various points in history.
- Attempt to structure the field of mathematics.
- Polarizations in mathematics.
- Who is the best mathematician?
- An exciting recent discovery?
- Why should we fear it?

C. A mystery theorem

- What is a graph?
- What is a tree?
- A theorem on flowers and trees.
- Topological invariants from local invariants.
- Gauss Bonnet Bonnet 1825 (Gauss) 1848 (Bonnet) 1888 (Dieck)

A quiz

10 questions. For distance learners, please submit within 24 hours to knill@math.harvard.edu. I will acknowledge your email.
In the same way as one has distinguished the canons of rhetorics: memory, invention, delivery, style, and arrangement, or combined the trivium: grammar, logic and rhetorics, with the quadrivium arithmetic, geometry, music, and astronomy, to get the seven liberal arts and sciences, one has also tried to organize all mathematical activities.

Historically, one has distinguished eight ancient roots of mathematics. These are 8 activities, each of which suggest a key area in mathematics:

- counting and sorting
- positioning and locating
- surveying and angulating
- balancing and weighing
- moving and hitting
- guessing and judging
- collecting and ordering
- analyzing and planning

Let's call this more modern adaptation the 12 modern roots of Mathematics:

- counting and sorting
- positioning and locating
- dividing and comparing
- balancing and weighing
- moving and hitting
- guessing and judging
- collecting and ordering
- slicing and stacking
- operating and memorizing
- optimizing and planning
- manipulating and solving
- analyzing and planning

Arithmetic
Geometry
Number theory
Algebra
Calculus
Set Theory
Probability
Topology
Analysis
Numerics
Dynamics
Algorithms

What are hot spots in mathematics today? Michael Atiyah identified in the year 2000 the following 6 hot spots in the development of mathematics:

- Number theory
- Algebraic geometry
- Stastics
- Dynamics
- Probability
- Algorithms

Also this choice is of course highly personal. One can easily add an other 12 of such polarizing quantities which help to distinguish or parametrize different parts of mathematical areas, especially the ambivalent pairs which are "hot":

<table>
<thead>
<tr>
<th>regularity and randomness</th>
<th>discrete and continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>integrable and non-integrable</td>
<td>existence and construction</td>
</tr>
<tr>
<td>invariants and perturbations</td>
<td>finite and infinite dimensional</td>
</tr>
<tr>
<td>experimental and deductive</td>
<td>topological and dimensional</td>
</tr>
<tr>
<td>polynomial and exponential</td>
<td>practical and theoretical</td>
</tr>
<tr>
<td>applied and abstract</td>
<td>axiomatic and case based</td>
</tr>
</tbody>
</table>

An other possibility to refine the fields of mathematics is to combine different of the 12 areas. Examples are probabilistic number theory, algebraic geometry, numerical analysis, geometric number theory, numerical algebra, algebraic topology, geometric probability, algebraic number theory, dynamical probability = stochastic processes. Almost every pair is an actual field. Finally, lets give a short answer to the question: What is Mathematics?

Mathematics is the science of structure.

The goal is to illustrate some of these structures from a historical point of view.