Lecture 13: Experimental mathematics

Experimental mathematics brings mathematics close to physics: we do experiments, for example with the help of a computer. Instead of talking about this abstractly, I tell you a concrete story which illustrates this.

Benfords law tells that the first digits of the sequence $2^n$ has a distribution which satisfies $p_k = \log_{10}(1 + 1/k)$. The digit 1 for example occurs with about 30 percent. During our discussion, we asked ourselves, what happens if we look at the first significant digit of $n^2$ or the first significant digit of the primes $p_n$.

First experiment: exponentials

We look at the numbers $2^n$ for $n = 1$ to $n = 1000000$ and look at the first digit:

For factorials, the limiting distribution is known to be the Benford distribution. There is no reason why $\log_{10}(n!) \mod 1$ should not be uniformly distributed.

Second experiment: squares

Now let’s look at the first significant digit of the squares $1^2, 4, 9, 1, 2, 9, 6, 8, 1, 1, 1, 1, 2, 2, 3, 3, 4, 4$.

Third experiment: Primes

What is the first significant digit of the prime numbers?

Forth experiment: factorials

For factorials, the limiting distribution is known to be the Benford distribution. There is no reason why $\log_{10}(n!) \mod 1$ should not be uniformly distributed.
Fifth experiment: partitions

Also for the partition numbers, \( p(n) \), which give the number of possibilities in which the number \( n \) can be written as a sum of integers, we measure that the Benford distribution takes place. As far as we know this is not known.

```math
\text{data} = \text{Table}[
\hspace{1em} \text{First}[\text{IntegerDigits}[\text{PartitionsP}[n]]], \{n, 1, 10000\};
\text{S = Histogram[}\text{data}, 10, \text{ColorFunction} \rightarrow \text{Hue}]\]
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Problems with known answers

1. Find the distribution of the first significant digit of \( n^2 + 2^n \).
2. Find the distribution of the first significant digit of the Fibonacci sequence 1,1,2,3,5,8,13,....
3. Find the distribution of the first significant digit of numbers \( n^{100} \).
4. Verify that for the factorials \( n! \), the first significant digit has the Benford distribution? (Benford)
5. Verify that for \( n^n \), the first significant digit has the Benford distribution? ¹

Problems with unknown answers

6. Find the distribution of the first significant digit of primes. One finds a \( n \) dependent generalized Benford law with \( \alpha(n) = 1/(\log(n) - 1.1) \). ²
7. Find the distribution of the first significant digit of \( \lfloor \exp(\sqrt{n}) \rfloor \) where where \( \lfloor x \rfloor \) is the largest integer smaller or equal to \( x \).
8. What is the distribution of the first significant digit of \( \lfloor n \log(n) \rfloor \) where \( \lfloor x \rfloor \) is the largest integer smaller or equal to \( x \).
9. What is the distribution of the first significant digit of the Partition numbers \( p(n) \)?
10. What is the distribution of the first significant digit of \( \lfloor n^2 \sin(n) \rfloor \) where \( \lfloor x \rfloor \) is the largest integer smaller or equal to \( x \).