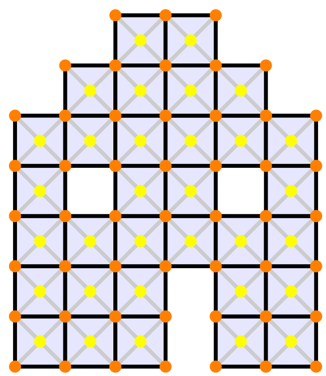
A mathematical theorem

Objective

In this worksheet, we work on a particular result in mathematics. Our task is to understand the theorem, place it into the landscape of mathematics and gain insight why the theorem is true.

Bricks, Windows, Houses and Towns

A unit square is called a **brick**. We draw the diagonals and call their intersection the **center** of the brick. Each brick has **4 corners**. If two bricks have a common corner, they **touch**. Bricks with two common corners **face.** Their intersection is then called a **face**. Two bricks are **connected**, if you can go from one brick to the other by crossing faces between other bricks in the house. A **house** is a finite union of bricks such that any two bricks in the house are connected. A **town** is a finite union of houses. A brick-free region which is enclosed by bricks of the house is called a **window** of the house.

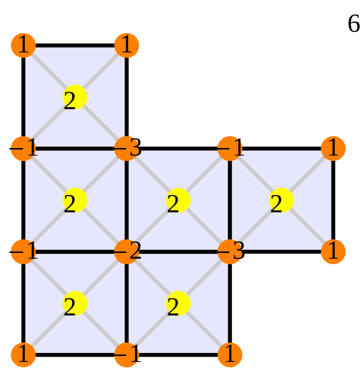


Rules

We assign numbers called **curvatures** to all the nodes of a house. Here are the rules:

- At an intersection of 4 bricks meet, put $\boxed{-2}$.
- At an edge where 2 bricks meet, put $\boxed{-1}$.
- At an inner corner, where 3 bricks meet, put $\boxed{-3}$
- At an outer corner, where only 1 brick is, put $\boxed{1}$
- At a center of a brick, put $\boxed{2}$.

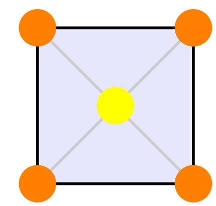
The sum of all these curvatures is called the **total curvature** of the house or town.



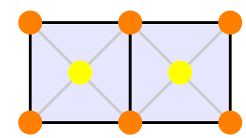
Examples of houses

Lets add up some curvatures:

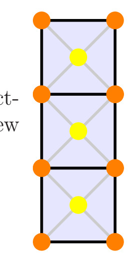
1. For one brick alone, we have 2 in the center and 1 at each corner. This adds up to 6.



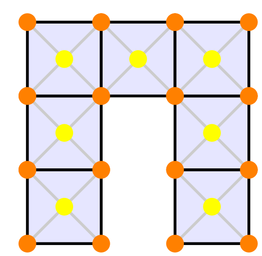
2. If two bricks face each other, we have 2 centers. This is 4. We have 2 flat corners, this adds -2 so that we are down to 2. Now we have 4 corners and this adds up to 6.



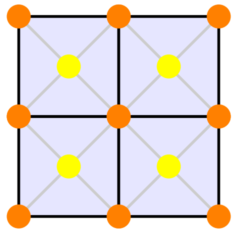
3. Assume now, three bricks forming a rectangle of size 3×1. We have added a new center and 2 flat corners. We still have 6.



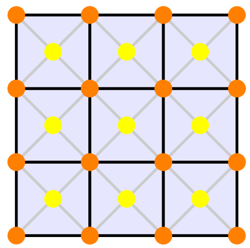
4. Eight bricks form an arch.



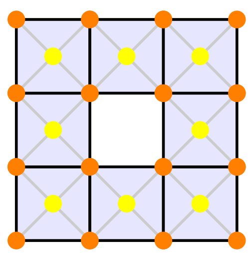
5. Four bricks call fill a 2×2 square. We go from experiment 4) and add a center (+2), and an outer corner (+1). Additionally, we have converted a corner with 3 bricks to a corner with 4 bricks (+1) and transformed two outer corners to edges (-4)



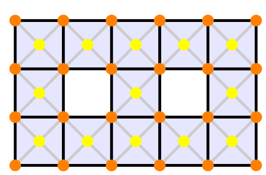
6. Lets finally look at a house with 9 bricks in a 3×3 square.



7. Lets add a window. What happens with the total curvature?



8. Here is a house with two windows.



The theorem

For a house G, define $\chi(G) = (2h - 2g)$, where h is the number of houses and g is the total number of windows. It is called the **Euler characteristic** of the town. The number of houses and the total number of windows determines the total curvature of the town:

Theorem: For any town G, the total curvature is equal to $3\chi(G) = 6(h - w)$.

To the proof:
1. Because curvatures and χ of individual houses add up, we can concentrate on one house.
2. Add bricks to each window until each window is minimal and consists of only one missing brick. Each of these "home improvements" not change the number of windows nor the total curvature.
3. Now fill in a brick into a 1-brick chamber, then the total curvature increases by 6, the total curvature of a single brick.
4. Once we have a house with no windows, we can remove bricks from the outside until we have only one brick left.

Explanations

To the Euler characteristic: The Euler characteristic is equal to $V - E + F$ where V is the number of **nodes** (either centers of bricks or corners of bricks), E is the number of **edges** (connections between nodes) and F is the number of **faces**, triangles enclosed by edges. The number of windows is also called **genus** in mathematics and denoted g. We have proven here the relation ship $\chi = 6 - 6g$ for houses.
To the curvature: For every point in this house, now look at the circle of radius 1 meaning all the adjacent points and count the number N of steps you have to do in this circle. If the circle around a point x is closed, define $K(x) = 6 - N(x)$ and otherwise define $K(x) = 3 - N(x)$.
To the theorem: We have seen a discrete version of a famous theorem called **Gauss-Bonnet theorem**. It relates local data with global data. The global data do not change if the house is deformed. As long as the total number of windows and houses is the same, the Euler characteristic is the same. Only the **topology** of the town matters. Results like that are at the heart of **topology**. The result is graph theoretical because it is true for any graph which is **"two dimensional with boundary"**. See http://www.math.harvard.edu/ knill/graphgeometry or google the term "discrete Gauss-Bonnet".
Historical: The theorem belongs to different fields of mathematics. It illustrates a topic in **topology**. It also belongs to **graph theory**. Because the geometric concept of "curvature" is involved, also is part of **geometry**. The **Gauss Bonnet** theorem is a corner-stone of **differential geometry** which deals with also with **surfaces**. At every point of surface, one can assign a number called **curvature** which tells about how the surface is bent.

A sphere has positive curvature everywhere. A doughnut has negative curvature in the inner part and positive curvature on the outer part. Integrating up the curvature over the entire surface is a multiple of the Euler characteristic of the surface. The Euler characteristic of a surface is $2h - 2g$ where h is the number of connected components and g is the number of "holes" in the surface. If the surface is the "town", the individual components are the "houses" and the "holes" are the "windows".