Lecture 2: Solutions and Remarks

1. Approximation

We derived in the first worksheet the Dirichlet theorem from Minkowski’s theorem. Here are some remarks to the solution.

a) The region is symmetric. The region is a parallelogram which matches itself after turning by 180 degrees. The region is also convex. Any parallelogram is convex.

b) The area is 4. Note that we have to make the parallelogram a tiny bit larger if we want to have an area larger than 4. What happens in Minkowski’s theorem with regions which have nice boundaries that the result is still true for area 4 but the lattice point can be at the boundary.

c) Yes, there is a lattice point in the region or at the boundary of the region. The distance \( |\alpha p - q| \) has an absolute value smaller or equal than \( 1/n \).

d) Divide \( |\alpha p - q| \leq 1/n \) by \( p \) to get \( |\alpha - p/q| \leq 1/(nq) \).

e) Yes, chose any large \( n \). With the \( p/q \) provided by the result, we have \( |\alpha - p/q| \leq 1/(nq) \) but this implies also \( |\alpha - p/q| \leq 1/q^2 \).

2. Approximation postscriptum

Here is a direct proof of Dirichlet’s theorem: by the pigeonhole principle: at least one of the \( n \) intervals \([k/n, (k+1)/n]\) in \([0,1]\) contains two elements of the set \( \{0, \{\alpha\}, \{2\alpha\}, \ldots \{n\alpha\}\} \), where \( \{ka\} \) is the fractional part of \( ka \). So \( |(ka - ka) + p| \leq 1/n \) for some integer \( p \) and \( q = k - l < n \), Division through \( q \) gives \( |\alpha + p/q| \leq 1/(nq) \).

3. Irrationality

Here are the solutions to the second worksheet

a) The proof that \( \sqrt{6} \) is irrational is similar. Assume \( \sqrt{6} = p/q \). This implies \( 6q^2 = p^2 \) which means that \( p \) must be even. But then also \( q \) must be even since 6 contains only one factor 2 and \( p^2 \) is divisible by 4.

b) Yes. If \( \sqrt{\alpha} = p/q \) were rational, then also \( x = p^2/q^2 \) would be rational.

c) Yes. Assume \( (\sqrt{2} + \sqrt{3}) = p/q \). Squaring this gives \( 2 + 3 + \sqrt{6} = p^2/q^2 \) and so \( \sqrt{6} = p^2/q^2 - 5 \). But we have seen in a) that \( \sqrt{6} \) is irrational.

d) Yes. If it were rational, then \( \sqrt{5} \) would be rational and the irrationality of \( \sqrt{7} \) can be established in the same way as the irrationality of \( \sqrt{2} \).

e) Also this proof can be done in the same way as the prototype. If \( \log_{10}(7) = p/q \) then \( 7 = 10^{p/q} \) and \( 7^q = 10^p \) but the right hand side is even, while the left hand side is odd.

4. Irrationality postscriptum

Are there two irrational numbers \( a, b \) such that \( a^b \) is rational? Here is a nonconstructive answer: either \( a = \sqrt{2}, b = \sqrt{2} \) is a solution or \( a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2} \) is a solution.

Proof. If the first case is not true, then we can use \( a = \sqrt{2^{\sqrt{2}}}, b = \sqrt{2} \) as an example.

The funny thing is that we do not know whether \( \sqrt{2^{\sqrt{2}}} \) is rational or not. I learned about this proof first from Gerhard Jaeger at ETH (now at the University in Bern, Switzerland), while I have been a course assistant in the mathematics department. Theoretical computer scientists are especially interested in the nature of such nonconstructive proofs. By the way, this example can be found in the Princeton Companion of Mathematics, at the very beginning of part III or [?] page 28.

The question whether there is an \( \alpha \) such that \( \alpha^\alpha \) is irrational is more subtle. I only came up with a proof which uses some calculus: the function \( f(x) = x^x \) is monotone and continuous on \( (1/2,1) \) would be mapped into the set of irrational numbers. Continuity prevents that. While this proof looks nonconstructive, it allows to construct numbers: just take a rational like 4/5 in the image of \( f \) and take the irrational \( \alpha = f^{-1}(4/5) \).

5. Writing numbers

Here are the solutions to the third worksheet:

1. \( 129 = \text{CXXVIII} \)

2. \( 6432 = F \text{u}\lambda \beta \).

3. \( 1000 = 16 \cdot 60 + 40 \). The Transliteration is 16,40.

4. There are 4 lotus flowers, which gives 40’000. Then there are 6 monkey tails which is 600. Then there are 2 tens and 2 ones. In total, the number is 40’622.

5. \( 441 = 20^2 + 2 \cdot 20 + 1 \) which has the transliteration 1, 2, 1. The solution looks similar than the one provided in the example for 445. But instead of a bar (5) we only have a dot.