1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 28: Second Hourly Practice

Welcome to the second hourly. It will take place on April 9, 2019 at 9:00 AM sharp in Hall D. Please fill out your name in the box above.

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. We want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is also space on the back of each page.
- If you finish a problem somewhere else, please indicate on the problem page so that we find it.
- You have 75 minutes for this hourly.

PROBLEMS

Problem 28P.1 (10 points):
a) Prove, using one of the theorems we have seen in the course that any matrix of the form $A - A^T$ is diagonalizable if A is an arbitrary real $n \times n$ matrix.
b) Prove, using one of the theorems we have seen in the course that any matrix of the form AA^T is diagonalizable if A is an arbitrary real $n \times n$ matrix.
c) The matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has the characteristic polynomial $-\lambda^5 + 1 = 0$. What does the theorem of Cayley-Hamilton state?

Problem 28P.2 (10 points):
Find the characteristic polynomial and the eigenvalues of the following matrices:

a) $\begin{bmatrix} -1 & 3 \\ 4 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 & 3 \\ 1 & 4 & 3 \\ 3 & 4 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$.

Problem 28P.3 (10 points):
Check the boxes which apply for all matrices of the type:

	invertible	diagonalizable	symmetric	real eigenvalues
Projection matrix				
Shear matrix				
Rotation matrix				
Reflection matrix				
$A^2 = 0, A \neq 0$				
Diagonal matrix				

Problem 28P.4 (10 points, each sub problem is 2 points):
a) (5 points) Find the determinant of the following matrix. You have to give reasoning!

$$A = \begin{bmatrix} 22 & 2 & 2 & 2 & 2 \\ 2 & 22 & 2 & 2 & 2 \\ 2 & 2 & 22 & 2 & 2 \\ 2 & 2 & 2 & 22 & 2 \\ 2 & 2 & 2 & 2 & 22 \end{bmatrix}$$

b) (5 points) Find an eigenbasis of A. It does not have to be orthonormal.

Problem 28P.5 (10 points):
Find the possibly complex eigenvalues for the following matrices:
a) (2 points)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

b) (2 points)

$$B = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

c) (2 points)

$$C = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

d) (2 points)

$$D = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

e) (2 points)

$$E = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix} .$$

Problem 28P.6 (10 points):
Find a closed-form solution of the recursion

$$x_{n+1} = 3x_n - 2x_{n-1}$$

with $x(1) = 1, x(0) = 4$ and determine with the system is stable. Make sure to write this first as a discrete dynamical system $(x(t+1), x(t)) = A(x(t), x(t-1))$ then use the initial condition $[1, 4]$.

Problem 28P.7 (10 points):
When we try to find a closed-form solution of the system

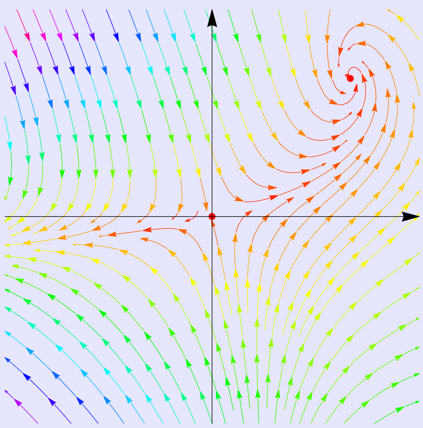
$$\begin{aligned} x' &= x - 2y \\ y' &= 2x - 3y \end{aligned}$$

with $x(0) = 2, y(0) = 1$ we run into trouble. Outline why, then tell how we still can find a closed-form solution. You don't have to do that explicitly. Just show to which matrix A of the system is similar to. What we want to know is whether the system is stable.

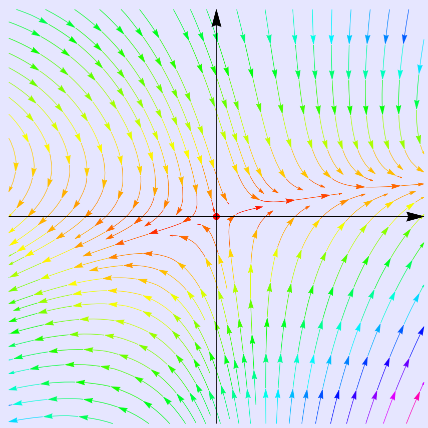
Problem 28P.8 (10 points):
We consider the nonlinear system of differential equations

$$\begin{aligned} \frac{d}{dt}x &= x + y - xy \\ \frac{d}{dt}y &= x - 3y + xy \;. \end{aligned}$$

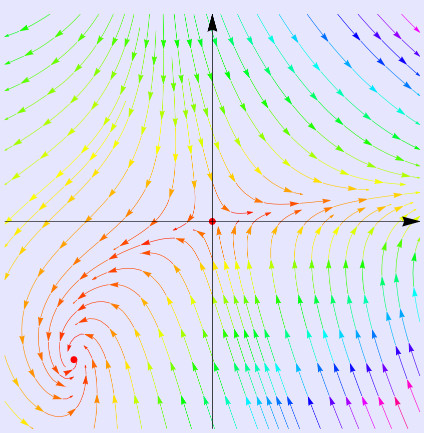
a) (2 points) Find the equilibrium points.
b) (3 points) Find the Jacobian matrix at each equilibrium point.
c) (3 points) Use the Jacobean matrix at an equilibrium to determine for each equilibrium point whether it is stable or not.
d) (2 points) Which of the diagrams A-D is the phase portrait of the system above?



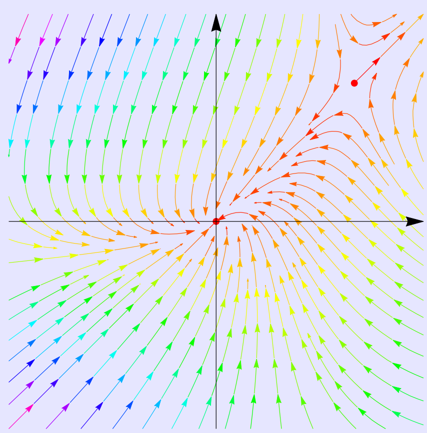
A



B



C



D

Problem 28P.9 (10 points):
(4 points) a) (5 points) Assume T is a transformation on $C^\infty(\mathbb{T})$, the linear space of 2π-periodic functions on the real line. Which transformations are linear?

Transformation	Check if linear
$Tf(x) = f(x+1)$	
$Tf(x) = f(\cos(x))$	
$Tf(x) = f'(x+\cos(x))$	
$Tf(x) = f(f(x)\cos(x))$	
$Tf(x) = \cos(f(x))$	
$Tf(x) = \cos(x) + f(x)$	

(5 points) The rest are knowledge questions which do not need any reasoning.
a) Which mathematician was a hermite in the later part of his life?
b) What is the entropy of the map $T(x) = 22x$ on $\mathbb{R}^1/\mathbb{Z}^1$.
c) Assume a 5×5 matrix has 2 Jordan blocks. Is it diagonalizable?
d) Find the eigenvector of the eigenvalue problem $Df = 3f$.
e) How big is the dimension of the solution space $(D^5 + D^3 + D)f = 0$?

Problem 28P.10 (10 points):
Find the general solution to the following differential equations:
a) (1 point)
$$f'(t) = 1/(t+1)$$
b) (1 point)
$$f''(t) = e^t + t$$
c) (2 points)
$$f''(t) + f(t) = t + 2$$
d) (2 points)
$$f''(t) - 2f'(t) + f(t) = e^t$$
e) (2 points)
$$f''(t) - f(t) = e^t + \sin(t)$$
f) (2 points)
$$f''(t) - f(t) = e^{-3t}$$