Unit 25: Function Spaces

Lecture

25.1. We have worked so far with $M(n,m)$, the linear space of all $n \times m$ matrices and especially with the Euclidean space $\mathbb{R}^n = M(n,1)$. When working with differential equations, it is necessary to work also with **spaces of functions**. Like vectors, functions can be added, scaled and contain a zero element, the function which is constant 0. From now on, when we speak about a linear space, we mean an **abstract linear space**, a set X which we can add, scale and have a zero element.

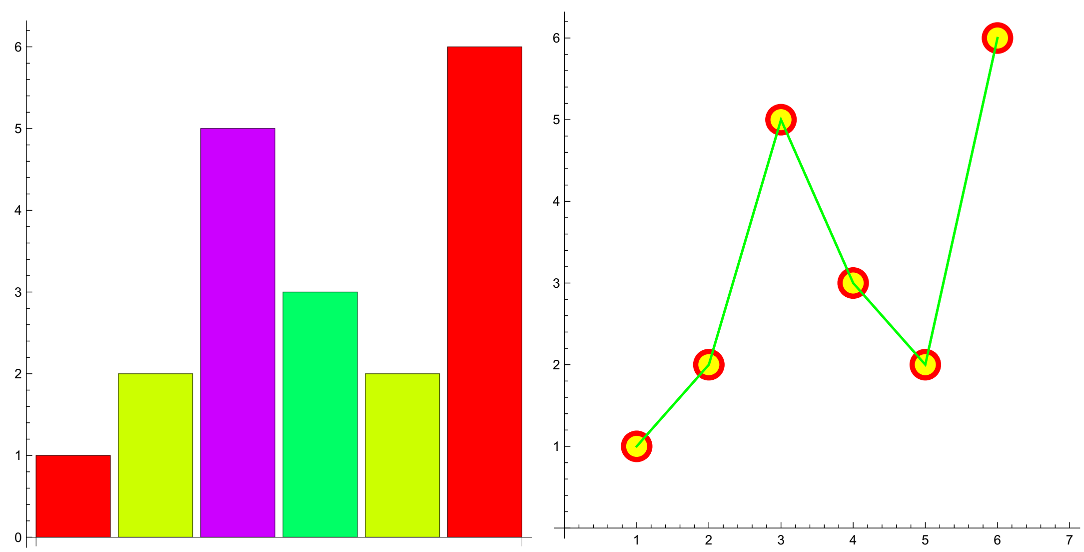

Figure 1. The vector $[1,2,5,3,2,6]$ can be interpreted as a function: just define $f(k) = v_k$. The index k is the input and the output is the value of the function is v_k. By looking at the function, we can visualize and see this 6-dimensional vector.

25.2. The space $C(\mathbb{R})$ of all continuous functions is a linear space. It contains vectors like $f(x) = \sin(x), g(x) = x^3 + 1$ or $h(x) = \exp(x)$. Functions can be added like $(f+h)(x) = \sin(x) + e^x$, they can be scaled like $(7f)(x) = 7\sin(x)$. Any function space also needs to contain the **zero function** $0(x) = 0$ satisfies $(f+0)(x) = f(x)$.

25.3. Why do we want to look at spaces of functions? One of the main reasons for us here is that **solutions spaces of linear systems of differential equations are function spaces.** Another reason is that in probability theory, **random variables** are elements in function spaces. In physics, fields are function spaces. This includes scalar fields, vector fields, wave functions or parametrizations for describing geometric objects like surfaces or curves. Finally, **functions** are a universal language to describe **data**. In figure 1 we see a data set with 6 points. We can not draw the vector in $\mathbb{R}^6$ but we can draw the bar-chart. With many data points, a bar chart can look like the graph of a function. It is a function on the set $\{1, 2, 3, 4, 5, 6\}$ anf $v_k = f(k)$.

25.4. Here is a general principle to generate linear spaces:

Principle: If X is a set, all maps from X to $\mathbb{R}^m$ form a linear space.

25.5. For example, if $X = \{1, 2, 3\}$ then the set of all maps from X to $\mathbb{R}$ is equivalent to $\mathbb{R}^n$. With $[f(1), f(2), f(3)]^T$ we get a vector. If $X = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ then the set of all maps from X to $\mathbb{R}$ is equivalent to $M(2, 2)$. With $\begin{bmatrix} f(1,1) & f(1,2) \\ f(2,1) & f(2,2) \end{bmatrix}$ we get a matrix. If $X = \mathbb{R}$, we get the set of all maps from X to $\mathbb{R}$. It is a large infinite dimensional space. If $X = \mathbb{R}^2$, we get the set of all functions $f(x, y)$ of two variables. The space of all maps from $\mathbb{R}$ to $\mathbb{R}^2$ is the space of all **parametrized planar curves**. The space of all maps from $\mathbb{R}^2$ to $\mathbb{R}^3$ is the space of all **parametrized surfaces**.

25.6. We can select subspaces of function spaces. For example, the space $C(\mathbb{R})$ of **continuous functions** contains the space $C^1(\mathbb{R})$ of all **differentiable functions** or the space $C^\infty(\mathbb{R})$ of all **smooth functions** or the space $P(\mathbb{R})$ of polynomials. It is convenient to look at $P_n(\mathbb{R})$, the space of all polynomials of degree $\leq n$. Also the space $C^\infty(\mathbb{R}, \mathbb{R}^3)$ of all smooth parametrized curves in space is a linear space. Another important space is $C^\infty(\mathbb{T})$ of 2π-periodic smooth functions. They can be seen as functions on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, which is the line in which all points in distance 2π are identified.

25.7. Let us look at the space $P_n = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\}$ of polynomials of degree $\leq n$.

Principle: The space P_n is a linear space of dimension $n + 1$.

Proof. It is a linear space because we can add such functions, scale them and there is the zero function $f(x) = 0$. The functions $\mathcal{B} = \{1, x, x^2, x^3, \ldots, x^n\}$ form a basis. First of all, the set $\mathcal{B}$ spans the space P_n. To see that the set is linearly independent assume that $f(x) = a_0 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$. By evaluating at $x = 0$, we see $a_0 = 0$. By looking at $f'(0) = 0$, we see that $a_1 = 0$, by looking at $f''(0) = 0$ we see $a_2 = 0$. Continue in the same way and compute the n'th derivative to see $a_n = 0$. ☐

25.8. As in the space of Euclidean spaces, we can find new linear spaces by looking at the kernel or the image of some transformation T. The most important transformation for us is the **derivative map** $T(f) = f'$. We call it D. So, $D \sin = \cos$ and $Dx^5 = 5x^4$.

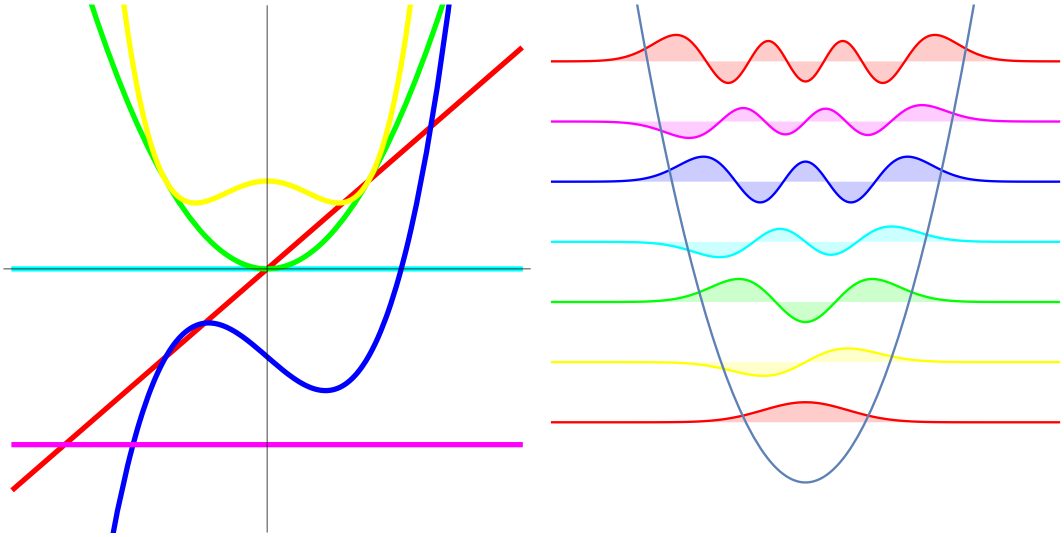

FIGURE 2. Left: graphs of 5 polynomials $f_0(x) = 0$, $f_1(x) = -2$, $f_2(x) = x$, $f_3(x) := x^2$, $f_4(x) := x^3 - x - 1$, $f_5(x) = x^4 - x^2 + 1$. The function $f_0(x)$ is the zero function. The functions $\{f_1, f_2, f_3, f_4, f_5\}$ form a basis of P_4. Right: graphs of eigenfunctions $f_n(x)$ of the harmonic oscillator operator $T = -D^2 + x^2$. The graphs are lifted to have average $(2n + 1)$, the n'th energy level.

Theorem: Kernel and image of a linear transformation are linear spaces.

Proof. Let $X = \ker(T)$. To verify that X is a linear space, we check three things: (i) if x, y are in X, then $x + y$ is in X. Proof: If $T(x) = 0, T(y) = 0$, then $T(x + y) = T(x) + T(y) = 0 + 0 = 0$. (ii) if x is in X, then λx in X. Proof: If $T(x) = 0$, then $T(\lambda x) = \lambda T(x) = \lambda 0 = 0$. (iii) We have 0 in X. Proof: $T(0) = 0$. □

25.9. What is the kernel and image of the transformation $Df = f'$ on $C^\infty(\mathbb{R})$? To find the kernel, we look at all functions f which satisfy $Df = f' = 0$. By integration, we see $f = c$ is a constant. So, the nullity of D is 1:

Principle: The kernel $\ker(D) = \{c \mid c \text{ is real}\}$ is one-dimensional.

25.10. To find the image, we want to see which functions f can be reached as $f = Dg$. Given f, we can form $g(x) = \int_0^x f(t)\, dt$. By the fundamental theorem of calculus, we see $Dg = g' = f(x)$.

Principle: The image of D is the entire space $\text{im}(D) = C^\infty$.

25.11. In the next lecture we will learn how to find solutions to differential equations like $f''(x) + 3f'(x) + 2f(x) = 0$. We will write this as an equation $(D^2 + 3D + 2)f = 0$ which means that the solution is the kernel of a transformation $T = D^2 + 3D + 2$. Now, because this is $(D + 2)(D + 1)f = 0$. Solutions can now be obtained by looking at $(D + 1)f = 0$ and $(D + 2)f = 0$, which has solutions $C_1 e^{-x}$ and $C_2 e^{-2x}$. So, the general solution is $f(x) = C_1 e^{-x} + C_2 e^{-2x}$.

Homework

This homework is due on Tuesday, 4/09/2019.

Problem 25.1: Which spaces X are linear spaces?
a) All polynomials of degree 2 or 3.
b) All smooth functions with $f'(1) = 0$.
c) All continuous periodic functions $f(x+1) = f(x)$ with $f(0) = 1$.
d) All functions satisfying $f''(x) - f(x) = 0$.
e) All smooth functions with $\lim_{|x|\to\infty} f'(x) = 0$.
f) All continuous real valued function $f(x, y, z)$ of three variables.
g) All continuous vector fields $F(x, y) = [P(x, y), Q(x, y)]$.
h) All parametrizations $r(t) = r(t + 2\pi) = [x(t), y(t), z(t)]$.
i) All curves $r(t) = [x(t), y(t)]$ in the plane which pass through $(1, 1)$.
j) All 4K movies, maps from $[0, 1]$ to $M(3200, 2400)$.

Problem 25.2: A polynomial $p(x, y)$ is of degree n, if the largest term $a_{kl}x^k x^l$ satisfies $k + l = n$. For example, $f(x, y) = 3x^4y^5 + xy + 3$ has degree 9. a) What is the dimension of the set of polynomials of degree less than 3? b) write down a basis. c) find a formula for the dimension of the space of all polynomials of degree n?

Problem 25.3: The linear map $Df(x) = f'(x)$ is an example of a **differential operator**. As it has a kernel, there is no unique inverse. One inverse is $Sf(x) = D^{-1}f(x) = \int_0^x f(t)\, dt$.
a) Evaluate $D\sin, D\cos$, $D\tan$, $S1/(1 + x^2)$, $S\tan$.
b) Find an eigenfunction f of D to the eigenvalue -22.
c) Verify that if f is an eigenfunction of D to the eigenvalue 2, then f is also an eigenfunction of $D^4 - 2D + 22$. What is the eigenvalue?

Problem 25.4: a) Find a basis for the kernel of D^3 on the linear space P of polynomials.
b) Find the image $D^3 + D + 1$ on the linear space P. c) Find the kernel of $Af = (D - \sin(t))f(t)$ on $C^\infty(\mathbb{T})$.

Problem 25.5: a) Solve $D^3f = 0$ with the additional condition $f(0) = 3, f'(0) = 1, f''(0) = 2$. b) Solve $D^3f = \cos(x)$ with $f(0) = 3, f'(0) = 1, f''(0) = 2$.

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