Unit 22: Stability

Lecture

22.1. A linear dynamical system is either a discrete time dynamical system \( x(t+1) = Ax(t) \) or a continuous time dynamical systems \( x'(t) = Ax(t) \). It is called asymptotically stable if for all initial conditions \( x(0) \), the orbit \( x(t) \) converges to the origin 0 as \( t \to \infty \). The one-dimensional case is clear: the discrete time system \( x(t+1) = \lambda x(t) \) has the solution \( x(t) = \lambda^t x(0) \) and is asymptotically stable if and only if \( |\lambda| < 1 \). The continuous time system \( x'(t) = \lambda x(t) \) has the solution \( x(t) = e^{\lambda t} x(0) \). This is asymptotically stable if and only if \( \lambda < 0 \). If \( \lambda = a + ib \), then \( e^{(a+ib)t} = e^{at} e^{ibt} \) shows that asymptotic stability happens if \( \text{Re}(\lambda) = a < 0 \).

22.2. Let us first discuss the stability for discrete dynamical systems \( x(t+1) = Ax(t) \), where \( A \) is a \( n \times n \) matrix.

22.3. Theorem: A discrete dynamical system \( x(t+1) = Ax(t) \) is asymptotically stable if and only if all eigenvalues of \( A \) satisfy \( |\lambda_j| < 1 \).

Proof. (i) If \( A \) has an eigenbasis, this follows from the closed-form solution: assume \( ||\lambda_k|| \leq \lambda < 1 \). From \( |x(t)| \leq |c_1||\lambda_1|^t + \cdots + |c_n||\lambda_n|^t \leq (\sum_i |c_i||\lambda|^t) \), we see that the solution approaches 0 exponentially fast.

(ii) The general case needs the Jordan normal form theorem proven below which tells that every matrix \( A \) can be conjugated to \( B + N \), where \( B \) is the diagonal matrix containing the eigenvalues and \( N^n = 0 \). We have now \( (B + N)^t = B^t + B(n,1)B^{t-1}N + \cdots + B(n,n)B^{t-n}N^{n-1} \), where \( B(n,k) \) are the Binomial coefficients. The eigenvalues of \( A \) are the same as the eigenvalues of \( B \). By (i), we have \( B^t \to 0 \). So, also \( A^t \to 0 \). □

22.4. In the case of continuous time dynamical system \( x'(t) = Ax(t) \), the complex eigenvalues will later play an important role but they are also important for discrete dynamical systems.

22.5. Theorem: A continuous dynamical system is asymptotically stable if and only if all eigenvalues satisfy \( \text{Re}(\lambda_j) < 0 \).
Proof. We can see this as a discrete time dynamical system with time step \( U = e^A \) because the solution \( e^{At} \) can be written as \( U^t \). We need therefore that \( e^{\lambda_j(U)} < 1 \). Which is equivalent to \( \text{Re}(\lambda_j) < 0 \). \( \square \)

22.6. A \( m \times m \) matrix \( J \) is a Jordan block, if \( Je_1 = \lambda e_1 \), and \( Je_k = \lambda e_k + e_{k+1} \) for \( k = 2, \ldots, m \). A matrix is \( A \) in Jordan normal form if it is block diagonal, where each block is a Jordan block. The shear matrix \( J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) is an example of a \( 2 \times 2 \) Jordan block.

**Theorem:** Every \( A \in M(n,n) \) is similar to \( B \in M(n,n) \) in Jordan normal form.

Proof. A vector \( v \) is called a generalized eigenvector of the eigenvalue \( \lambda \) of \( A \) if \( (A - \lambda)^m v = 0 \) for some \( m > 0 \). The smallest integer \( m \) for which the relation holds is called the eigenvector rank of \( v \). If the eigenvector rank is 1, then \( v \) is an actual eigenvector.

22.7. Take an eigenvalue \( \lambda \) of \( A \) and pick the maximal \( m \) for which there is a generalized eigenvector of rank \( m \). This means that there is a vector \( v_m \) such that \( (A - \lambda)^m v_m = 0 \) but \( (A - \lambda)^{m-1} v_m \neq 0 \). Now define \( v_k = (A - \lambda)^{m-k} v_m \). The vector \( v_1 = (A - \lambda)^{m-1} v_m \) is an eigenvector of \( A \) because \( (A - \lambda)v_1 = 0 \). Because \( (A - \lambda)v_{k+1} = v_k \) and \( (A - \lambda)v_1 = 0 \), on the space \( V \) spanned by \( B = \{v_1, v_2, \ldots, v_m\} \), the matrix is given by a Jordan block with 0 in the diagonal. This space \( V \) is called a generalized eigenspace of \( A \). It is left invariant by \( A \).

22.8. Take a generalized eigenspace \( V \) defined by an eigenvector \( v_1 \) and a generalized eigenspace \( W \) defined by another eigenvector \( w_1 \) not parallel to \( v_1 \). Claim: \( V \) and \( W \) have no common vector. Proof: Assume \( x = \sum_{i=1}^m a_i v_i = \sum_{j=1}^l b_j w_j \). Then \( (A - \lambda)x = \sum_{i=1}^m a_i v_{i-1} = \sum_{j=1}^l b_j w_{j-1} \). If \( m \neq l \), and say \( m < l \) then applying \( (A - \lambda) \) \( m - 1 \) times gives \( a_m v_1 = b_{l-m} w_{l-m} \) but since \( v_1 \) is an eigenvector and \( v_{l-m} \) is not, this does not work. If \( m = l \), then we end up with \( a_m v_1 = b_m w_1 \) which is not possible as \( v_1 \) and \( w_1 \) are not parallel unless \( a_m = b_m = 0 \). Now repeat the same argument but only apply \( m - 2 \) times. This gives \( a_{m-1} v_{m-1} = b_{m-1} w_{m-1} \) but this implies \( v_{m-1}, w_{m-1} \) are parallel showing again \( v_m \) is parallel to \( w_m \). We get \( a_{m-1} = b_{m-1} = 0 \) and eventually see that all \( a_k = b_k = 0 \).

22.9. The proof of the theorem for \( n \times n \) matrices uses induction with respect to \( n \). The case \( n = 1 \) is clear as a \( 1 \times 1 \) matrix is already in Jordan normal form. Now assume that for every \( k < n \), every \( k \times k \) matrix is similar to a matrix in Jordan normal form. Take a generalized eigenvector \( v \) and build the Jordan normal block acting on the generalized eigenspace \( V \). By the previous paragraph, we can find a basis such that \( V^\perp \) is invariant. Using induction, there is a Jordan decomposition for \( A \) acting on the \( V^\perp \). The matrix \( A \) has in this bases now a Jordan decomposition with an additional block. \( \square \)
22.10. This implies the **Cayley-Hamilton theorem**:  

**Theorem:** If \( p \) is the characteristic polynomial of \( A \), then \( p(A) = 0 \).

*Proof.* It is enough to show this for a matrix in Jordan normal form for which the characteristic polynomial is \( \lambda^m \). But \( A^m = 0 \). \( \square \)

**Examples**

22.11. You have a **checking account** of one thousand dollars and a **savings account** with one thousand dollars. Every day, you pay 0.001 percent to the bank for both accounts. But the deal is sweet: your checking account will get every day 1000 times the amount in the savings account. Will you get rich?

\[
\begin{align*}
C(t+1) &= 0.999c_n + 1000s_n \\
S(t+1) &= 0.999s_n
\end{align*}
\]

We will discuss this in class.

22.12. For which constants \( a \) is the system \( x(t+1) = Ax(t) \) stable?

a) \( A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \), b) \( A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} \), c) \( A = \begin{bmatrix} a & a \\ a & a \end{bmatrix} \).

**Solution.**

a) The trace is zero, the determinant is \(-a^2\). We have stability if \(|a| < 1\). You can also see this from the eigenvalues, \( a, -a \).

b) Look at the trace-determinant plane. The trace is \( a \), the determinant \(-1\). This is nowhere inside the stability triangle so that the system is always unstable.

c) The eigenvalues are \( 0, 2a \). The system is stable if and only if \(|2a| < 1\) which means \(|a| < 1/2\).

22.13. In two dimensions, we can see asymptotic stability from the trace and determinant. The reason is that the characteristic polynomial and so the eigenvalues only need the trace and determinant.

A two dimensional discrete dynamical system has asymptotic stability if and only if \((\text{tr}(A), \text{det}(A))\) is contained in the interior of the **stability triangle** bounded by the lines \(\text{det}(A) = 1, \text{det}(A) = \text{tr}(A) - 1\) and \(\text{det}(A) = -\text{tr}(A) - 1\). 

*Proof.* Write \( T = \text{tr}(A)/2, D = \text{det}(A) \). If \(|D| \geq 1\), there is no asymptotic stability. If \( \lambda = T + \sqrt{T^2 - D} = \pm 1 \), then \( T^2 - D = (\pm 1 - T)^2 \) and \( D = 1 \pm 2T \). For \( D \leq -1 + |2T| \) we have a real eigenvalue \( \geq 1 \). The conditions for stability is therefore \( D > |2T| - 1 \). It implies automatically \( D > -1 \) so that the triangle can be described shortly as \([\text{tr}(A)] - 1 < \text{det}(A) < 1\).

For a two-dimensional continuous dynamical system we have asymptotic stability if and only if \(\text{tr}(A) < 0\) and \(\text{det}(A) > 0\).
Problem 22.1: Determine whether the matrix is stable for the discrete dynamical system or for the continuous dynamical system or for both: a) \[ A = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}, \] b) \[ B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \] c) \[ C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}. \]

Problem 22.2: True or false? We say \( A \) is stable if the origin \( \vec{0} \) is asymptotically stable for \( x(t+1) = A(x(t)) \). Give short explanations:

a) 1 is stable. 
b) 0 matrix is stable.
c) a horizontal shear is stable 
d) a reflection matrix is stable.
e) \( A \) is stable if and only if \( A^T \) is stable.
f) \( A \) is stable if and only if \( A^{-1} \) is stable.
g) \( A \) is stable if and only if \( A + 1 \) is stable.
h) \( A \) is stable if and only if \( A^2 \) is stable.
i) \( A \) is stable if \( A^2 = 0 \).
j) \( A \) is unstable if \( A^2 = A \).
k) \( A \) is stable if \( A \) is diagonalizable.

Problem 22.3: a) Check the Cayley-Hamilton theorem for the matrix \( A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \). b) Ditto for a rotation dilation matrix defined by \( a, b \).

Problem 22.4: For which real values \( k \) does the drawing rule  
\[ x(t+1) = x(t) - ky(t) \]
\[ y(t+1) = y(t) + kx(t+1) \]
produce trajectories which are ellipses? Write first the system as a discrete dynamical system \( v(t+1) = Av(t) \), then look for \( k \) values for which the eigenvalues satisfy \( |\lambda_k| = 1 \).

Problem 22.5: Find the eigenvalues of 
\[ A = \begin{bmatrix} 0 & a & b & c & 0 & 0 \\ 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & a & b & c \\ c & 0 & 0 & 0 & a & b \\ b & c & 0 & 0 & 0 & a \\ a & b & c & 0 & 0 & 0 \end{bmatrix} \]
Where \( a, b \) and \( c \) are arbitrary constants. Verify that the discrete dynamical system is stable for \( |a| + |b| + |c| < 1 \).