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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 13: First Hourly Practice

Welcome to the first hourly. It will take place on February 26, 2019 at 9:00 AM sharp in Hall D. You can already fill out your name in the box above.

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. We want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is also space on the back of each page.
- If you finish a problem somewhere else, please indicate on the problem page so that we find it.
- You have 75 minutes for this hourly.

PROBLEMS

Problem 13P.1 (10 points):
We prove $\text{im}(A) = \text{im}(AA^T)$ in two steps. Do them both.
a) Prove that $\text{im}(A)$ contains $\text{im}(AA^T)$.
b) Prove that $\text{im}(AA^T)$ contains $\text{im}(A)$.

Problem 13P.2 (10 points):
Decide in each case whether the set X is a linear space. If it is, prove it, if it is not, then give a reason why it is not.
a) The space of 4×4 matrices with zero trace.
b) The space of 4×4 quaternion matrices.
c) The space of 2×2 matrices λQ, where Q is an orthogonal matrix and λ is real.
d) The space of 3×3 matrices λQ, where Q is an orthogonal matrix and λ is real.
e) The space of 5×5 matrices with entries 0 or 1.
f) The image $\text{im}(1)$ of the identity 3×3 matrix 1.
g) The kernel of the matrix $A = [1, 2, 3, 4]$.
h) The set of all vectors in $\mathbb{R}^3$ for which $x^2 + y^2 + z^2 = 0$.
i) The set of all vectors in $\mathbb{R}^3$ for which $xyz = 0$.
j) The set of 3×3 matrices with orthogonal columns.

Problem 13P.3 (10 points):
a) (2 points) Is the transformation $T(x,y) = (x^2, y^2)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ linear?
b) (2 points) Is the map $T(A) = \text{tr}(A)1$ as a map from $M(3,3)$ to $M(3,3)$ linear? Here 1 is the identity matrix.
c) (4 points) Match the transformation type:
A: **rotation dilation**, B: **reflection dilation**, C: **shear dilation**, D: **projection dilation**

Fill A-D				
Matrix	$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ 4 & -4 \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

d) (4 points) SO, SU or NO (no SO nor SU)? Yo, you have to decide so!

SO,SU,NO				
Matrix	$\begin{bmatrix} i & -i \\ i & i \end{bmatrix}$	$\begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} i & i \\ i & i \end{bmatrix}$	$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Problem 13P.4 (10 points, each sub problem is 2 points):
a) Which physicist promoted the use of examples to understand a theory?
b) We describe a general linear map T from $M(2,2)$ to $M(2,2)$. In a suitable basis, is this map given by a 2×2 matrix or a 4×4 matrix?
c) Is it true that $f(x) = e^{3x}$ satisfies $f(x) = O(e^x)$?
d) Give an example of a 4×4 matrix A for which $\ker(A) = \text{im}(A)$.
e) If $B = S^{-1}AS$, what can you say about the determinants of A and B?

Problem 13P.5 (10 points):
In the **checkers matrix**, the entry 1 means that the checkers initial condition has a checker piece there and 0 means that that field is empty:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

a) (6 points) Find a basis for the kernel of A.
b) (4 points) Find a basis for the image of A.

Problem 13P.6 (10 points):

a) (5 points) The projection-dilation matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ in the basis

$$\mathcal{B} = \{v_1, v_2, v_3\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$$

is given by a matrix B. Find this 3×3 matrix B.
b) (5 points) A linear transformation T satisfies

$$T(v_1) = v_2, T(v_2) = v_3, T(v_3) = v_1$$

where v_1, v_2, v_3 are given in a). Find the matrix R implementing this transformation in the standard basis.

Problem 13P.7 (10 points):
Find the QR decomposition of the following matrices

a)(2 points) $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

b) (2 points) $B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 1 \\ 3 & 1 \\ 1 & -3 \end{bmatrix}$.

c) (2 points)$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

d) (2 points)$D = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

e) (2 points) $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem 13P.8 (10 points):

a) (2 points) Find the determinant of the "prime" matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 13 & 0 & 0 & 11 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}.$$

b) (2 points) Find the determinant of the "count to 12" matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix}.$$

c) (2 points) Find the determinant of the "2-1" matrix

$$C = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

d) (2 points) Find the determinant of the "Pascal triangle" matrix

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}.$$

e) (2 points) Find the determinant of the "mystery" matrix:

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 1 & 4 & 4 & 4 \\ 1 & 1 & 1 & 5 & 5 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix}.$$

Problem 13P.9 (10 points):
Find the function

$$a|x| - b|x-1| = y$$

which is the best fit for the data

x	y
1	2
3	2
-2	1
2	0

Problem 13P.10 (10 points):
The matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

are called the **Gamma matrices** or **Dirac matrices**.
a) (4 points) Are the columns orthonormal? That is, is it true that $A^*A = 1$, where $A^* = \overline{A}^T$.

	A	B	C	D
Orthonormal columns?				

b) (3 points) Compute A^2 and D^2 and $AD + DA$.
c) (3 points) Write down the inverse of A,B and D.