DEFINITION The transpose of a matrix $A$ is the matrix $(A^T)_{ij} = A_{ji}$. If $A$ is a $n \times m$ matrix, then $A^T$ is a $m \times n$ matrix. For square matrices, the transposed matrix is obtained by reflecting the matrix at the diagonal.

EXAMPLES The transpose of a vector $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the row vector $A^T = [1 \ 2 \ 3]$. The transpose of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

A PROPERTY OF THE TRANSPOSE. PROOFS. a) Because $x \cdot Ay = \sum_j \sum_i x_iA_{ij}y_j$ and $A^T x \cdot y = \sum_j \sum_i A_{ji}x_iy_j$, the two expressions are the same by renaming $i$ and $j$.

b) $(AB)^T = B^T A^T$.

c) $(A^T)^T = A$.

DEFINITION. A $n \times n$ matrix $A$ is called orthogonal if $A^T A = I$. The corresponding linear transformation is called orthogonal.

INVERSE. It is easy to invert an orthogonal matrix: $A^{-1} = A^T$.

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EXAMPLES. The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ is orthogonal because its column vectors have length 1 and are orthogonal to each other. Indeed: $A^T A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A reflection at a line is an orthogonal transformation because the columns of the matrix $A$ have length 1 and are orthogonal. Indeed: $A^T A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(2\phi) & -\sin(2\phi) \\ \sin(2\phi) & \cos(2\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

FACTS. An orthogonal transformation preserves the dot product: $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ Proof: this is a homework assignment: Hint: just look at the properties of the transpose.

Orthogonal transformations preserve the length of vectors as well as the angles between them.

Proof. We have $||A\vec{x}||^2 = A\vec{x} \cdot A\vec{x} = \vec{x} \cdot \vec{x} ||\vec{x}||^2$. Let $\alpha$ be the angle between $\vec{x}$ and $\vec{y}$ and let $\beta$ denote the angle between $A\vec{x}$ and $A\vec{y}$ and $\alpha$ the angle between $\vec{x}$ and $\vec{y}$. Using $A\vec{x} \cdot A\vec{y} = A\vec{x} \cdot \vec{y}$ we get $||A\vec{x}|| ||A\vec{y}|| \cos(\beta) = A\vec{x} \cdot A\vec{y} = A\vec{x} \cdot A\vec{y} = ||\vec{x}|| ||\vec{y}|| \cos(\alpha)$. Because $||A\vec{x}|| = ||\vec{x}||$, $||A\vec{y}|| = ||\vec{y}||$, this means $\cos(\alpha) = \cos(\beta)$. Because this property holds for all vectors we can rotate $\vec{x}$ in plane $V$ spanned by $\vec{x}$ and $\vec{y}$ by an angle $\phi$ to get $\cos(\alpha + \phi) = \cos(\beta + \phi)$ for all $\phi$. Differentiation with respect to $\phi$ at $\phi = 0$ shows also $\sin(\alpha) = \sin(\beta)$ so that $\alpha = \beta$.

ORTHOGONAL MATRICES AND BASIS. A linear transformation $A$ is orthogonal if and only if the column vectors of $A$ form an orthonormal basis. (That is what $A^T A = I$ means.)

COMPOSITION OF ORTHOGONAL TRANSFORMATIONS. The composition of two orthogonal transformations is orthogonal. The inverse of an orthogonal transformation is orthogonal. Proof. The properties of the transpose give $(AB)^T AB = B^T A^T AB = B^T B = I$ and $(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = I$.

EXAMPLES. The composition of two reflections at a line is a rotation. The composition of two rotations is a rotation. The composition of a reflections at a plane with a reflection at another plane is a rotation (the axis of rotation is the intersection of the planes).
ORTHOGONAL PROJECTIONS. The orthogonal projection $P$ onto a linear space with orthonormal basis $v_1, \ldots, v_n$ is the matrix $AA^T$, where $A$ is the matrix with column vectors $v_i$.

To see this just translate the formula $P \vec{x} = (v_1 \cdot \vec{x}) v_1 + \ldots + (v_n \cdot \vec{x}) v_n$ into the language of matrices: $A^T \vec{x}$ is a vector with components $b_i = (v_i \cdot \vec{x})$ and $A b$ is the sum of the $b_i v_i$, where $\vec{v}_i$ are the column vectors of $A$.

EXAMPLE. Find the orthogonal projection $P$ from $\mathbb{R}^3$ to the linear space spanned by $v_1 = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Solution: $AA^T = \begin{pmatrix} 0 & 1 & 4/5 \\ 3/5 & 0 & 0 \\ 4/5 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 9/25 & 12/25 \\ 0 & 12/25 & 16/25 \end{pmatrix}$.

WHY DO WE CARE ABOUT ORTHOGONAL TRANSFORMATIONS?

- In Physics, Galileo transformations are compositions of translations with orthogonal transformations. The laws of classical mechanics are invariant under such transformations. This is a symmetry.
- Many coordinate transformations are orthogonal transformations. We will see examples when dealing with differential equations.
- In the QR decomposition of a matrix $A$, the matrix $Q$ is orthogonal. Because $Q^{-1} = Q^t$, this allows to invert $A$ easier.
- Fourier transformations are orthogonal transformations. We will see this transformation later in the course. In application, it is useful in computer graphics (i.e. JPG), sound compression (i.e. MP3).
- Quantum mechanical evolutions (when written as real matrices) are orthogonal transformations.

WHICH OF THE FOLLOWING MAPS ARE ORTHOGONAL TRANSFORMATIONS?:

- Shear in the plane.
- Projection in three dimensions onto a plane.
- Reflection in two dimensions at the origin.
- Reflection in three dimensions at a plane.
- Dilation with factor 2.
- The Lorenz boost $x \mapsto A \vec{x}$ in the plane with $A = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}$.
- A translation.

CHANGING COORDINATES ON THE EARTH. Problem: what is the matrix which rotates a point on earth with (latitude,longitude)$=(a_1, b_1)$ to a point with (latitude,longitude)$=(a_2, b_2)$? Solution: The matrix which rotate the point $(0, 0)$ to $(a, b)$ a composition of two rotations. The first rotation brings the point into the right latitude, the second brings the point into the right longitude. $R_{a,b} = \begin{pmatrix} \cos(b) & -\sin(b) & 0 \\ \sin(b) & \cos(b) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $R_{\alpha,\beta} = \begin{pmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix}$. To bring a point $(a_1, b_1)$ to a point $(a_2, b_2)$, we form $A = R_{a_2,b_2} R_{a_1,b_1}^{-1}$.

Example: With Cambridge (USA): $(a_1, b_1) = (42.366944, 288.893889)\pi/180$ and Zürich (Switzerland): $(a_2, b_2) = (47.377778, 8.551111)\pi/180$, we get the matrix $A = \begin{pmatrix} 0.178313 & -0.980176 & -0.0863732 \\ -0.983567 & 0.180074 & -0.0129873 \\ 0.028284 & -0.082638 & 0.996178 \end{pmatrix}$.