Linear spaces, Sect. 9.1 4/25/2001 Math 21b, O. Knill

Homework for Friday April 27, 1,2,3,4,5,6 in handout (a copy is below)

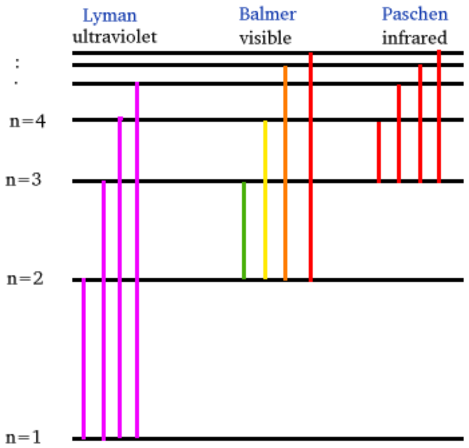
TO INIFINITY ... BUT WHY? The concept of a vector can be extended to infinite dimensions. If you look at functions, then one can add and scale them as vectors in Euclidean space. Linear maps between those spaces can be analized similarly as we analyzed finite dimensional maps. While a finite dimensional linear space is determined by its dimension, there is a wide variety of spaces in infinite dimensions. Nevertheless, many concepts from **linear algebra** extend to infinite dimensions. Why do we go through the trouble and look at more general linear spaces at all? Its because many laws in physics, economics or biology are described very well and effectively in that language. For example to describe the Hydrogen atom in quantum mechanics:

QUANTUM MECHANICS. A **quantum mechanical particle** is described by a complex valued function $\psi_t(x,y,z)$ having the property that $||\psi||^2 = \int |\psi_t(x,y,z)|^2 \; dxdydz = 1$. These "unit vectors" move according to a linear law: $\dot{\psi}_t = i\hbar L\psi$ called **Schrödinger equation**, where L is a linear map and $\hbar \sim 10^{-34} Js$ is the **Planck constant**. For a free particle $L\psi = -(\hbar^2/2m)\Delta\psi$ with $\Delta\psi = \psi_{xx} - \psi_y y - \psi_z z$. The integral $\int_\Omega |\psi_t(x,y,z)|^2 \; dxdydz$ is the probability to find the particle is in some region Ω. For a free particle, solutions are **de Broglie waves** $\psi_t(k) = e^{i(k\cdot x - \omega t)}\psi_0$, where $\omega = \hbar|k|^2/2m$. The wave describes a particle with mass m, energy $\hbar\omega = \hbar^2|k^2|/(2m)$ traveling with velocity $\hbar k$. The "length" $||\psi_t(k)||^2$ of $\psi_t(k)$ is not finite but by linearity one can add different solutions to obtain new waves which have length 1. If $\int \psi_0(k)g(k) \; dk = \psi_0$, then $\psi_t = \int \psi_t(k)g(k) \; dk$. Beeing forced to add waves with different velocities in order to describe a localized wave is related to the **uncertainty principle**. The last integral is a Fourier integral an is an infinite-dimensional "rotation" which swaps position and momentum. When particles move in an attractive field, the linear map L has eigenvalues and the possible energy states becomes "quantized:

HYDROGEN ATOM. If $L = -(\hbar^2/2m)\Delta - (4\pi\epsilon_0)^{-1}e^2/r$, meaning that $1/r\psi(x) = (1/||x||)\psi(x)$, then ψ_t describes an electron in the **Hydrogen atom**. The linear map L has eigenvalues $\lambda_n = E_n = -2\pi R_y/n^2$, where R_y is the **Rydberg constant** $3.29 * 10^{15} Hz$. There are different "eigenvectors" ψ_{nlm} to the eigenvalue E_n, where $l = 0, 1, ..., n-1$ $m = -l, ..., l$ are related to the **angular momentum** of the particle. The eigenvectors are functions: for $l = 0$, they are called the $s-$**states**, for $l = 1$, the $p-$**states** for $l = 2$ the $d-$**states**.

An electron jumping from a state n_1 to a state n_2 absorb the energy $\Delta E = E_{n_2} - E_{n_1} = 2\pi\hbar R_y(1/n_1^2 - 1/n_2^2)$ which corresponds to a frequency $\nu = \Delta E/(2\pi\hbar) = R_y(1/n_1^2 - 1/n_2^2)$. For example for $n_1 = 2, n_2 = 4$, we obtain $R_y(1/4 - 1/16) = 6.1710^{16 Hz}$ which corresponds to a wave length $4.8610^{-7} m = 486 nm \sim$ blue.

Lyman	$n_1 = 1, n_2 = 2, 3, \dots$	$\nu = R_y(1 - \frac{1}{n_2^2})$	Ultraviolet
Balmer	$n_1 = 2, n_2 = 3, 4, \dots$	$\nu = R_y(\frac{1}{4} - \frac{1}{n_2^2})$	Visible
Paschen	$n_1 = 3, n_2 = 4, 5, \dots$	$\nu = R_y(\frac{1}{9} - \frac{1}{n_2^2})$	Infrared
Brackett	$n_1 = 4, n_2 = 5, 6, \dots$	$\nu = R_y(\frac{1}{16} - \frac{1}{n_3^2}$	Infrared



PERIODIC SYSTEM OF ELEMENTS. With more nuclei in the center of the atom, different electrons occupy as fermions never the same state twice but fill up the possible energy states from below. This ultimately determines the **periodic system of elements**. Things are a bit more complicated but essentially, the **linear map** L defined on functions explaines completely, how the chemical elements are built!

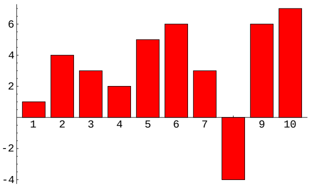
OTHER EXAMPLES. As a motivation for studing linear spaces in more generality, we could also have looked at examples in fluid dynamics, acoustics, problems in elasticity, theromdyamics or electricity, where partial differential equations describe the behavior of the system. In the case, when the differential equations are linear, linear algebra can help to describe it. This is not limited to chemistry, biology or physics: financial markets are studies with Brownian motion. Originally designed to study the random motion of particles in a fluid, Brownian motion can for example serve as a model for the fluctuations of market data. In probability theory **random variables** form a linear space. Measureing correlations is actually measuring angles between vectors.

TERMINOLOGY. The definition of a **linear space** = **vector space** is the same as in finite dimensions. It is a set on which we can add objects and scale objects by real scalars having all the usual rules for vectors satisfied: $f + (g + h) = (f + g) + h$, $\lambda(f + g) = \lambda f + \lambda g$, $f + g = g + f$ etc. A linear space has a zero element. Most notions introduced so far for vectors generalize unchanged. A linear space is **infinite dimensional** if it has no finite dimensional basis. A map T between linear spaces is linar if $T(f + g) = Tf + Tg$, $T\lambda f = \lambda Tf$. It has an eigenvalue λ to the eigenvector f if $T(f) = \lambda f$. If f is a function one calls it also an **eigenfunction**.
EXAMPLE. $T : f \mapsto -f''$ is a linear map on smooth 2π periodic functions. A constant function is in the kernal. If $\int_0^{2\pi} f(x)\, dx = 0$ then f is in the image. $f(x) = \sin(nx)$ is an eigenfunction to the eigenvalue n^2.

EXAMPLES OF LINEAR SPACES.

- $C(R)$ the set of **continuous functions** on R. Zero is the function $f(x) = 0$.
- $P(R)$ the set of **polynomials**. An example is $p(x) = 5 + x + x^2 - x^3 + 3x^4$.
- $C^\infty(R)$ the set of **smooth functions** on the real line. An example is e^{-x^2}.
- $L^2(R)$ the set of **square integrable functions**. These are functions satisfying $\int |f(x)|^2\, dx < \infty$ (f can be approximated by continuous functions f_n so that $d(f_n, f) = \int |f_n - f|^2\, dx$).
- The set of continuous **1 periodic functions**. Examples are $\sin(2\pi x)$.
- The set of all continuous functions on the interval $[0, 1]$ which satisfy $f(0) = f(1) = 0$.
- The set of all **sequences** x_n such that $\sum_n x_n^2 < \infty$. Examples are $x_n = 1/n$.
- The set of all **linear maps** from a linear space to itself. For example, all linear maps on the plane form a linear space with dimension 4. Zero is the transformation $T(f) = 0$.
- The set of all linear maps from $C(R)$ to R. For example $T(f) = f(0)$ is such a map.
- The **solution set of a linear differential equation** is a linear space.
- If $T : X \to Y$ is linear, then the **kernel** and the **image** of T are linear spaces.

FUNCTIONS - VECTORS? You should think of $f(x)$ as the coordinate of f and $f(x)$ as a coordinate of f. A 10 dimensional vector $v = [1, 4, 3, 2, 5, 6, 3, -4, 6, 7]$ can be drawn as a graph $k \mapsto v_k$. A function is an infinite dimensional version of this. Instead of finitely many numbers $v_1, v_2, \dots, v_n$, we need infinitely many numbers $f(x)$ where x runs over the reals.



LINEAR DESCRIPTION NONLINEAR MAPS. Nonlinear systems could be studied through linear systems: take $\dot{x} = x(1 - x)$ for example and define $x_n = x^n$. Now $\dot{x_1} = x_1 - x_2$, $\dot{x}_2 = 2x\dot{x} = 2x^2 = 2x^3 = 2x_2 - 2x_3$ etc. The vector $x = [x_1, x_2, \dots,]$ satisfies the differential equation $\dot{x} = Ax$, where A is a infinite matrix $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ \dots & \dots & \dots & \dots \end{bmatrix}$. This could be done also for the chaotic Lorentz system. The point is that linear maps in infinite dimensions can be complicated.

EXERCICES. (This is a copy from the PDF document)

1. Which of the following sets are linear subspaces of C^∞? Justify your answers:
(i) All continuous functions from R to R.
(ii) All $f \in C^\infty$ such that $f(0) + f'(0) = 0$.
(iii) All $f \in C^\infty$ such that $f + f' = 0$.
(iv) All $f \in C^\infty$ such that $f(0) = 1$.

2. Which of the following subsets of C^∞ are linearly independent? Justify your answers.
(i) $1, t, t^2, t^3$
(ii) $1 + t, 1 - t, t^2, 1 + t + t^2$.
(iii) $\sin(t), e^t, e^{-t}$.
(iv) $\sin(t), \cos(t), \sin(t + \pi/3)$

3. Which of the following maps are linear? Justify your answers.
(i) $T : C^\infty \to R, T(f) = f(0)$.
(ii) $T : C^\infty \to C^\infty, T(f) = f^2 + f'$.
(iii) $T : C^\infty \to R^2, T(f) = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$.
(iv) $T : C^\infty \to R$, $T(f) = \int_0^1 f(t)\, dt$.

4. Find a basis for the kernel of $T : C^\infty \to C^\infty$ given by $T(f)(t) = f''(t) - f(0)$.
5. Find a basis for the image of $T : C^\infty \to C^\infty$ given by $T(f)(t) = f(0) + f'(0)t + [f(0) + f'(0)]t^2$.
6) Find the eigenvalues and eigenspaces for $T : C^\infty \to C^\infty$ given by $T(f) = f + f'$.