Orthogonality 3/2/2001 **Math 21b, O. Knill**

ON THE RELEVANCE OF ORTHOGONALITY.

1) During the pyramid age in Egypt (year -2800 until -2300), the egyptions used ropes divided into length ratios 3 : 4 : 5 to build triangles. This allowed them to triangulate areas quite precisely: for example to build irrigation (needed because the Nile was shaping the land constantly) or to build the pyramids: For the **great pyramid at Giza** with a base length of 230 meters, the average error on each side is less then 20 cm, an error of less then 1/1000! A key to achieve this was **orthogonality**

2) It was during one of Thales (-624 until -548) journeys to Egypt that he used a geometrical trick to **measure the height** of the great pyramid. He measured the size of the shadow of the pyramid. Using a stick, he found the relation between the length of the stick and the length of its shadow. The same length ratio applies to the pyramid (**orthogonal** triangles). Thales found also that triangles inscribed into a circle and having as the base as the diameter must be orthogonal.

3) The Pythagoreans (-572 until -507) were very interested in the discovery that the squares of a right angle would add up as the Pythagoras theorem $a^2 + b^2 = c^2$ tells. They were however puzzled in assigning a length to the diagonal of a square.

4) Eratosthenes (-274 until 194) realized that while the sun rays were **orthogonal** to the ground in the town of Scene, this did no more do so at the town of Alexandria where they would hit the ground at 7.2 degrees). Because the distance was about 500 miles and 7.2 is 1/50 of 360 degrees, he measured the circumfence of the earth as 25'000 miles which is pretty close to its actual value.

5) Closely related to **orthogonality** is **parallelism**. For a long time mathematicians tried to prove Euclid's parallel axiom using other postulates of Euclid (-325 until -265). These attempts had to fail because there are geometries in which parallel lines always meet (like on the sphere) or geometries, where parallel lines never meet (the Poincaré half plane). Also these geometries can be studied using linear algebra. The geometry on the sphere with **rotations** the geometry of the Poincare half plane using complex 2×2 matrices.

6) The question, whether in reality the angles of a right triangle always add up to 180 degrees became a real issue after the discovery of geometries, in which the measurement depends on the position in space. This Riemannian geometry, founded 150 years ago, is the foundation of **general relativity** a theory describing gravity geometrically: the presence of mass bends space-time.

7) In **probability theory** the notions **independence** or **decorrelated** appear. For example, when throwing dice, the number shown by the first dice is independent and therefore decorrelated from the number shown by the second dice. Decorrelation is identical to **orthogonality**, when vectors are associated to the random variables.

8) In **quantum mechanics**, states of atoms are described by functions which can be viewed as vectors also. The states with energy $-E_B/n^2$ (where $E_B = 13.6eV$ is the Bohr energy) in a hydrogen atom. States in an atom are **orthogonal**. Two states of two different atoms which don't interact are **orthogonal**. One of the challenges in quantum computing, where the computation deals with qubits (=vectors) is that orthogonality is not preserved during the computation. Different states can interact. This coupling is called **decoherence**.

DEFINITION. Two vectors v and w are called **orthogonal** if $v \cdot w = 0$.
Examples.
1) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$ in the plane.
2) Both v and w are orthogonal to $v \times w$ in three dimensional space.

A vector is called a **unit vector** if its length $||v|| = \sqrt{v \cdot v}$ is 1.
A set of vectors $v_1, \dots, v_n$ are called **orthogonal** if they are pairwise orthogonal. They are called **orthonormal** if they have additionally length 1. A basis is called an **orthonormal basis** if they are are orthogonal and each vector has length 1.
Example. For an orthonormal basis, the matrix $A_{ij} = v_i \cdot v_j$ is the unit matrix.

FACTS. Orthogonal vectors are linearly independent. If we have n of them in $\mathbf{R}^n$, then they form a basis.
Proof. Multiplying a possible **linear relation** $a_1v_1 + \ldots + a_nv_n = 0$ with v_k gives $a_kv_k \cdot v_k = a_k||v_k||^2 = 0$ and so $a_k = 0$. n linear independent vectors in $\mathbf{R}^n$ automatically span the space. (See the last hour: independence means that the matrix with columns v_i has trivial kernel. The dimension of the image is therefore n by the dimension formula).

DEFINITIONS. A vector v is called **orthogonal** to a linear space V if v is orthogonal to every vector in V. The **orthogonal complement** of a linear space V is the set W of all vectors which are orthogonal to V. It forms a linear space because $v \cdot w_1 = 0, v \ldots w_2 = 0$ implies $v \cdot (w_1 + w_2) = 0$. The **orthogonal projection** onto a linear space V with orthonormal basis $v_1, \ldots v_n$ is the linear map $A(x) = proj_V(x) = (v_1 \cdot x)v_1 + \ldots + (v_n \cdot x)v_n$. The vector $x - \mathrm{proj}_V(x)$ is in the orthogonal complement of V.

x and y are always vectors in $\mathbf{R}^n$ and V is a linear subspace:
PYTHAGORAS: If x and y are orthogonal, then If Pythagoras holds, then x, y are orthogonal.

$$||x + y||^2 = ||x||^2 + ||y||^2 \; .$$

PROJECTIONS DO NOT INCREASE LENGTH:

$$||\mathrm{proj}_V(x)|| \leq ||x|| \; .$$

(Hint: use Pythagoras) If $||\mathrm{proj}_V(x)|| = ||x||$, then x is in V.

CAUCHY-SCHWARTZ INEQUALITY:

$$|x \cdot y| \leq ||x|| \, ||y|| \; .$$

(remember the formula for $x \cdot y$ with angle α?) If $|x \cdot y| = ||x||||y||$, then x and y are parallel.
TRIANGLE INEQUALITY:

$$||x + y|| \leq ||x|| + ||y||$$

because $(x + y) \cdot (x + y) = ||x||^2 + ||y||^2 + 2x \cdot y \leq ||x||^2 + ||y||^2 + 2||x||||y|| = (||x|| + ||y||)^2$,

DEFINITION. The **angle** between two vectors x, y is

$$\alpha = \arccos\left(\frac{x \cdot y}{||x||||y||}\right) \; .$$

Example. The angle between two orthogonal vectors is 90 degrees or 270 degrees. If x and y represent data showing the deviation from the mean, then $\frac{x \cdot y}{||x||||y||}$ is called the **correlation** of the data. The extremest positive correlation is when $x = y$.

A QUESTION.
Express the fact that x is in the kernel of a matrix A using orthogonality of row/column vectors in A.
ANSWER: $Ax = 0$ means that $w_k \cdot x = 0$ for every row vector w_k of $\mathbf{R}^n$.
REMARK. We will call later the matrix A^T, obtained by switching rows and columns of A the **transpose** of A. You see already that the image of A^T is orthogonal to the kernel of A.

A QUESTION: Find a basis for the orthogonal complement of the linear space V spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. and $\begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$.

ANSWER: The orthogonality of $\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$ to the two vectors means solving the linear system of equations $x + 2y + 3z + 4w = 0, 4x + 5y + 6z + 7w = 9$.
Can you find a matrix A such that the kernel of A is the orthogonal complement of V? This reduces the problem to an older problem.