Homework for Monday April 30, 1,2,3,4,5,6 in Section 10.1

LINEAR DIFFERENTIAL EQUATIONS. \( Df = f' \) is a linear map on smooth functions \( C^\infty \). Also \( G_h f = hf \) is a linear map. Composition and addition of linear maps gives new linear maps. For example \( T f = D^2 f + hf \) is a linear map. If \( T \) is obtained like this then \( T f = 0 \) is called a linear differential equation. For linear maps \( T = p(D) \) which are polynomials in \( D \), the problem \( T f = 0 \) or more generally the eigenvalue problem \( T f = \lambda f \) are differential equations with constant coefficients. The eigenspace to the eigenvalue \( \lambda \) is a linear space. Especially, the kernel of \( T \) is a linear space.

FINDING THE KERNEL OF A POLYNOMIAL IN \( D \). How do we find a basis for the kernel of \( T f = f'' + 2f' + f \)? The linear map \( T \) can be written as a polynomial in \( D \) which means \( T = D^2 - D - 2 = (D + 1)(D - 2) \). The kernel of \( T \) contains the kernel of \( D - \lambda \) which is one-dimensional and spanned by \( f_1 = e^{2x} \). The kernel of \( T = (D - 2)(D + 1) \) also contains the kernel of \( D + 1 \) which is spanned by \( f_2 = e^{-x} \). The kernel of \( T \) is therefore two dimensional and spanned by \( e^{2x} \) and \( e^{-x} \).

THEOREM: If \( T = p(D) = D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0 \) on \( C^\infty \) then \( \dim(\ker(T)) = n \).

PROOF. \( T = p(D) = \prod(D - \lambda_j) \), where \( \lambda_j \) are the roots of the polynomial \( p \). The kernel of \( T \) contains the kernel of \( D - \lambda \) which is spanned by \( f_j(t) = e^\lambda t \). In the case when we have a factor \( (D - \lambda_j)^k \) of \( T \), then we have to consider the kernel of \( (D - \lambda_j)^k \) which is \( q(t)e^{\lambda t} \), where \( q \) is a polynomial of degree \( k - 1 \). For example, the kernel of \( (D - 1)^3 \) consists of all functions \( (a + bt + ct^2)e^t \).

SECOND PROOF. Write this as \( ADF = A\hat{f} = 0 \), where \( A \) is a \( n \times n \) matrix and \( F = [f, f', \ldots, f^{(n-1)}]^T \), where \( f^{(k)} = D^k f \) is the \( k \)th derivative. The linear map \( T = AD \) acts on vectors of functions. If all eigenvalues \( \lambda_j \) of \( A \) are different (they are the same \( \lambda_j \) as before), then \( A \) can be diagonalized. Solving the diagonal case \( BD = 0 \) is easy. It has a \( n \) dimensional kernel of vectors \( F = [f_1, \ldots, f_n]^T \), where \( f_i(t) = t \). If \( B = SAS^{-1} \), and \( F \) is in the kernel of \( BD \), then \( SF \) is in the kernel of \( AD \).

REMARK. The result can be generalized to the case, when \( a_j \) are functions of \( t \). Especially, \( T f = g \) has a solution, when \( T \) is of the form above. It is important that the highest power \( D^n \) has nothing in front of it which can be zero for some \( t \). For example \( tDf = f' \) has no solution in \( C^\infty \), because we can not integrate \( e^t/t \).

WHY ARE WE INTERESTED IN THE KERNEL?

- Equations \( T f = 0 \) where \( T = p(D) \) form linear differential equations with constant coefficients and we want to understand all solutions. Such equations are called homogeneous. Solving means finding a basis of the kernel of \( T \). In the above example, a general solution of \( f'' + 2f' + f = 0 \) can be written as \( f(t) = a_1 f_1(t) + a_2 f_2(t) \). If we fix two values like \( f(0), f'(0) \) or \( f(0), f(1) \), the solution is unique.
- If we want to solve \( T f = g \), which corresponds to a inhomogeneous equation then \( T^{-1} \) is not unique because we have a kernel. If \( g \) is in the image of \( T \) we can find at least one solution \( f \). The general solution is then \( f + \ker(T) \). For example, for \( T = D^2 \), which has \( C^\infty \) as its image, we can find a solution to \( D^2 f = t^5 \) by integrating twice: \( f(t) = \frac{t^6}{20} \). The kernel of \( T \) consists of all linear functions \( at + b \). The general solution to \( D^2 = t^5 \) is \( at + b + t^5/20 \). The integration constants parameterize actually the kernel of a linear map.
- In order to find the eigenspace of \( T \) to the eigenvalue \( \lambda \) we have to find the kernel of \( T - \lambda \).

AN EIGENVALUE PROBLEM. If \( T \) is the linear map \( T f = f'' \), what are the eigenvalues and eigenvectors?

SOLUTION. \( T f = \lambda f \) means \( f''(x) = \lambda f(x) \). You remember that the solutions are all of the form \( f(x) = a \cos(\lambda t) + b \sin(\lambda t) \). The kernel of \( D^2 \) consists of all functions \( f(x) = ax + b \). A basis of the kernel are \( f_1(x) = 1, f_2(x) = x \). The kernel is two-dimensional.
ON A DIFFERENT SPACE. Let us look at $C^\infty(I)$ consisting of all functions on the interval $I = [0, \pi]$ for which $f(0) = 0$ and $f(\pi) = 0.$ Now, in order that $f(0) = 0$ and $f(\pi) = 0,$ we must have $\lambda = n$ and $a = 0.$ The linear map $T$ has the eigenfunctions $f_n(x) = \sin(nx)$ to the eigenvalues $\lambda_n = -n^2.$ The kernel of $T$ is now trivial because there is no nonzero function $f$ which satisfy $Tf = 0.$

INTERPRETATION. For the eigenvalue problem $Tf = \lambda f$ on $C^\infty(I),$ the numbers $\lambda$ are the possible frequencies of the standing wave which is kept fixed at $0$ and $\pi.$

QUANTUM MECHANICAL INTERPRETATION. The problem $Tf = \lambda f$ on $C^\infty(I)$ describes the quantum mechanical particle in a box $[0, \pi].$ While $P = i\hbar D$ is the linear map representing the momentum, $H = P^2/2m - (\hbar^2/2m)D^2$ represents the kinetic energy. (In the case of the Hydrogen atom, we had additionally the energy of the $y$ and $z$ direction as well as the potential energy). The functions $\sin(nx)$ are eigenfunctions of $-D^2$ to the eigenvalue $\lambda_n = n^2,$ where $n = 1, 2, 3, \ldots.$ Therefore, $E_n = 2m n^2/\hbar^2$ are the eigenvalues of $H.$

These are the possible energies of the particle in the box. The quantized appearance of the energies is the origin for the name "quantum mechanics." If a particle is represented by $f_n = \sqrt{2/\pi} \sin(nx),$ which is normalized so that $\int_0^\pi f_n^2 \, dx = 1,$ then $f_n^2(x)$ is a probability density. The probability to find a particle with energy $2m n^2/\hbar^2$ in an interval $[a, b]$ is $\frac{2}{\pi} \int_a^b \sin^2(nx) \, dx.$ Max Planck (on left picture) had been forced to consider a discrete energy spectrum in order to explain the blackbody radiation.

Probability distribution in $n = 1$ Probability distribution in $n = 2$ Probability distribution in $n = 3$

MOTIVATION: WAVES. If we bend a string located on the graph of a function $x \mapsto T(x)$ on $[0, \pi]$ satisfying $T(0) = T(\pi) = 0,$ then the force $F(x)$ which pulls it back at the point $x$ is proportional to $T''.$ The string $T(x, t)$ satisfies $\ddot{T}(x, t) = c^2 T''(x, t),$ where $c$ is a constant. If we write $T(x, t) = u(x)v(t),$ then $\ddot{u} = \ddot{v}$ and $T'' = uv''.$ The equation becomes now $\ddot{u} = \ddot{v}$ or $\ddot{u}/(\ddot{v}) = u''/u.$ Because the left hand side depends only on $t$ and the right hand side only on $x,$ we have $\ddot{v}/(\ddot{v}) = u''/u = -n^2 = \text{const}.$ The right equation is an eigenvalue problem $\ddot{v} = \lambda u$ which has solutions for $\lambda = -n^2.$ The eigenvectors are $u_n(x) = \sin(nx).$ Now, $v_n(t) = \exp(inct)$ solves $\ddot{v} = -c^2 n^2 v$ so that $T_n(x, t) = u_n(x)v_n(t) = \sin(nx) \exp(inct)$ are solutions of the wave equation. General solutions can be obtained by taking superpositions of these waves $T(x, t) = \sum_n c_n \sin(nx) \exp(inct).$ The coefficients $c_n = a_n + ib_n$ are obtained from $T(x, 0) = \sum_n a_n \sin(nx)$ and $\dot{T}(x, 0) = \sum_n b_n n c_n \sin(nx).$ These are Fourier series which we will look at in the next class.