EIGENVALUES AND EIGENVECTORS.
A nonzero vector \( v \) is called an \textbf{eigenvector} of \( A \) if \( Av = \lambda v \) for some number \( \lambda \) which is called an \textbf{eigenvalue}.

EXAMPLES.
- \( v \) is an eigenvector to the eigenvalue 0 if and only if it is in the kernel of \( A \).
- If \( v \) is an eigenvector to the eigenvalue 1, then \( Av = v \). Example: a vector in the axis of rotation of a rotation \( A \). - If \( A \) is a diagonal matrix with diagonal elements \( a_i \), then the basis vectors \( e_i \) are eigenvectors.
- A shear \( A \) in the direction \( v \) has an eigenvector \( v \).
- A rotation in the plane by an angle 30 degrees has no eigenvector. (There are actually eigenvectors but they are complex).

LINEAR DYNAMICAL SYSTEMS.
Iterating a linear map \( x \mapsto Ax \) appears in many applications. One wants to understand what happens with \( x_1 = Ax, x_2 = A^2x, x_3 = A^3x, \ldots \).

One dimension: \( x \mapsto ax \) or \( x_{n+1} = ax_n \) has the solution \( x_n = a^n x_0 \). For example, \( 1.03^{20} \cdot 1000 = 1806.11 \) is the balance on a bank account which had 1000 dollars 20 years ago and if the interest rate was constant 3 percent.

In many cases the behavior of \( u_{n+1} \) does not only depend on \( u_n \) but also on \( u_{n-1} \) or earlier times. In that case we write \( (x_n, y_n) = (u_n, u_{n-1}) \) and get a linear map. \( x_{n+1}, y_{n+1} \) depend in a linear way on \( x_n, y_n \). We see two examples below.

LINEAR RECURSION PROBLEM: (from quantum mechanics) How does \( u_n \) grow if \( u_{n+1} + u_{n-1} = u_n \) and \( u_0 = 0, u_1 = 1 \). Plot \( (x_n, y_n) = (u_n, u_{n-1}) \) for different initial conditions like for example \( (2, 0), (0, 4) \).

(The picture shows electron diffraction patterns calculated using \textbf{Bloch waves}. By the way, the \( u_n \) just calculated is an example of a Bloch wave in a one dimensional crystal). Picture Source: P. Stadelman, 1995, cimesg1.epfl.ch/CIOL/asu94/ICT5.html)
LINEAR RECURSION PROBLEM:

A problem in the third section of Liber abbaci, published in 1202 by Leonardo Fibonacci (1170-1250):

A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

How does $u_n$ grow, if $u_{n+1} = u_n + u_{n-1}$? Plot $(x_n, y_n) = (u_n, u_{n-1})$ for different initial conditions like $(1, 0)$ or $(0, 1)$ and also $n < 0$.

WHERE DO LINEAR DYNAMICAL SYSTEMS APPEAR?

Linear systems $x \mapsto Ax$ appear in many places, like quantum mechanics, chaos theory, probability theory economics or biology. More examples are still to come.

EXAMPLE 1: Quantum mechanics. Some quantum mechanical systems of a particle in a potential $V$ are described by $(Lu)_n = u_{n+1} + u_{n-1} + V_n u_n$. Energies $E$ for which $(Lu)_n = Eu_n$, we have the recursion $u_{n+1} + u_{n-1} = (E - V_n)u_n$, when the potential is periodic in $n$, then this leads to a linear recursion problem. For example, if $V_n = V$ is constant, then $u_{n+1} + u_{n-1} = (E - V)u_n$. A question is for which $E$ the solutions stay bounded. You have seen above the case $E - V = 1$.

EXAMPLE 2: Chaos theory. In plasma physics, one studies maps like $(x,y) \mapsto (2x - y - a \sin(x), x)$. You see that $(0,0)$ is a fixed point. Near that fixed point, the map is described by its Jacobian $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2 - a & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. For which $a$ is this linear system stable near $(0,0)$ in the sense that a point near $(0,0)$ stays there.

EXAMPLE 3: Evolution of quantities. Example could be market systems, population quantities of different species, or ingredient quantities in a chemical reaction. A linear description might not always be a good model but it has the advantage that we can solve the system explicitly. Eigenvectors will provide the key to do so.

EXAMPLE 4: Markov Processes. The percentage of people using Apple OS or the Gnu/linux operating system is represented by a vector $\begin{bmatrix} m \\ l \end{bmatrix}$. Let 2/3 be the percentage of Mac OS users, who switch to Linux each month and 1/2 the percentage of Linux OS users, who switch to Apple. We look at the matrix $P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$. (It is called a Markov matrix: the entries satisfy $0 \leq P_{ij} \leq 1$ and the sum of each column elements is equal to 1). What ratio of Apple/Linux users do we have? We can simulate this as follows with a dice: start in the state $M=(1,0)$ (Mac). In the state $M$, if 3,4,5 or 6 shows up, switch to L=(0,1). Otherwise, keep M. If in the state L, if 1,2 or 3 shows up, switch to M otherwise keep L. The matrix $P$ has an eigenvector $(3/7, 4/7)$ to the eigenvalue 1. The interpretation is that eventually, the probability to be in state M is 3/7 and the probability to be in state L is 4/7.