AIM. We want explicit formulas for the inverse of a matrix $A$ or the solution $x$ of a linear equation $Ax = b$. While writing this in terms of determinants is not the most efficient way to compute these things, such formulas are useful for example when having parameters in the matrix. A symbolic algebra program can then for example give explicit formulas for the dependence of the solution $x$ on external parameters and allow theoretical predictions.

REMINDERS.
- An orthonormal matrix satisfies $Q^TQ = 1$. From this we get $|\det(Q)| = 1$.
- The determinant of a tridiagonal matrix is the product of the diagonal entries.
- Every matrix can be written as $A = QR$, where $Q$ is orthogonal and $R$ is upper tridiagonal.
- The image of the unit cube under a linear map $x \mapsto Ax$ is a parallelepiped $E_n$ spanned by the column vectors $v_1, \ldots, v_n$ of $A$.

VOLUME OF A PARALLELEPIPED. A $j$-dimensional parallelepiped $E_j$ spanned by vectors $v_1, \ldots, v_j$ has a $j - 1$ dimensional parallelepiped $E_{j-1}$ in the base. $E_{j-1}$ is contained in the vector space $V_{j-1}$ spanned by $v_1, \ldots, v_{j-1}$. The opposite "face" is in distance $|u_j| = |v_j - \proj_{V_{j-1}}v_j|$. The volume $\vol(E_j)$ satisfies $\vol(E_{j-1})|u_j|$.

ORIENTATION. Determinants allow us to define the orientation of $n$ vectors in $n$-dimensional space, (where we don’t have a "right hand rule" in general ...). Just look at the matrix $A$ with column vectors $v_j$ and define the orientation as the sign of $\det(A)$. In three dimensions, this agrees with the right hand rule: if $v_1$ is the thumb, $v_2$ is the pointing finger and $v_3$ is the middle finger, then their orientation is positive.

DETERMINANT AND VOLUME. The absolute value of the determinant of a $n \times n$ matrix $A$ is the volume of the $n$-dimensional parallelepiped $E_n$ spanned by the column vectors $v_j$ of $A$.

Proof. Use the QR decomposition $A = QR$, where $Q$ is orthogonal and $R$ is upper triangular. From $QQ^T = 1$, we get $1 = \det(Q)\det(Q^T) = \det(Q)^2$ see that $|\det(Q)| = 1$. Therefore, $\det(A) = \det(R)$. The determinant of $R$ is the product of the $|u_j| = |v_j - \proj_{V_{j-1}}v_j|$ which was the distance from $v_j$ to $V_{j-1}$ and height $\vol(E_{j-1})|u_j| |u_j|$. Inductively $\vol(E_j) = |u_j| |\vol(E_{j-1})|$ and therefore $\vol(E) = \prod_{j=1}^n |u_j| = \det(R)$.

MORE GENERALLY: The volume of a $k$ dimensional parallelepiped defined by the vectors $v_1, \ldots, v_k$ is $\sqrt{\det(A^T A)}$ because $A^T A = (QR)^T (QR) = R^T R$ and $\det(R^T R) = \det(R)^2 = (\prod_{j=1}^n |u_j|)^2$.

CHANGE OF VARIABLES. (For people who heard multi-variable calculus) If $x \mapsto y = u(x)$ is a change of variable, then the matrix $Du(x)$ is the linearisation of the map near $x$ and $|dy| = \|\det(Du(x))\| \cdot |dx|$. This leads to the change of variable formula

$$\int_S f(x) \, dx = \int_{u(S)} f(y) |\det(Du^{-1})(y)| \, dy$$

which can be remembered like in the 1 dimensional case:

**Example 1:** (1 dim) if $x = u^{-1}(y) = \sin(y)$, $dx = \cos(y) \, dy$. $\int_0^{\pi/2} \sqrt{1 - \sin^2(y)} \, dy = \int_0^{\pi/2} \cos(y) \, dy = \pi/4$.

**Example 2:** If $u(s,t) = (x(s,t), y(s,t), z(s,t))$ is a surface, then $A = Du(s,t)$ is a $3 \times 2$ matrix with column vectors $X = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix}$, $Y = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}$ (which are tangent vectors to the surface). Now

$$A^T A = \begin{bmatrix} X \cdot X & X \cdot Y \\ X \cdot Y & Y \cdot Y \end{bmatrix}$$

whose determinant is $||X||^2 ||Y||^2 - ||X \cdot Y||^2$ = $||X||^2 ||Y||^2 (1 - \cos(\phi))^2 = ||X||^2 ||Y||^2 \sin(\phi)^2$ = $||X \times Y||^2$. The expansion factor is $|X \times Y|$.
CRAMER'S RULE. This is an explicit formula for the solution of $Ax = b$. If $A_i$ is the matrix, where the column $v_i$ is replaced by $b$, then $x_i = \det(A_i)/\det(A)$.

Proof. $\det(A_i) = \det([v_1, \ldots, b, \ldots, v_n]) = \det([v_1, \ldots, (Ax), \ldots, v_n]) = \det([v_1, \ldots, \sum x_i v_i, \ldots, v_n]) = x_i \det([v_1, \ldots, v_n]) = x_i \det(A)$.

GABRIEL CRAMER. (1704-1752). Born in Geneva (Switzerland), he worked on geometry and analysis. He died during a trip to France, where he wanted to start retirement.

WHY IS CRAMERS RULE INTERESTING? Determining $x$ with these formulas is slower than with Gaussian elimination: a determinant calculation needs $n^3$ steps so that $n^3$ calculations are needed for the inverse via Cramer's rule. (Compare $n^3$ with Gaussian elimination).

The rule is important because if $A$ or $b$ depends on a parameter $\lambda$, and we want to see how $x$ depends on the parameter $\lambda$ one can find explicit formulas for $(d/d\lambda)x_i(\lambda)$.

Cramer's rule tells for example that the solution can depends in a sensitive way on parameters if the determinant is small (look at scissors).

EXAMPLE. In solid state physics, one is interested in the $\det(L - E)$, where

$$L = \begin{bmatrix} \lambda \cos(\alpha) & 1 & 0 & \cdots & 0 & 1 \\ 1 & \lambda \cos(2\alpha) & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & \cdots & 1 & \lambda \cos((n-1)\alpha) & 1 \\ 1 & 0 & \cdots & 0 & \lambda \cos(n\alpha) \end{bmatrix}$$

describes an electron in a periodic crystal, $E$ is the energy and $\alpha = 2\pi/n$. The electron can move (as a Bloch wave) whenever the determinant is negative. These intervals form the spectrum of the matrix. A physicist is interested for example in the dependence of the spectrum on the parameter $\lambda$ or $E$.

The graph shows the function $E \mapsto \log(|\det(L - E)|)$ in the case $\lambda = 2$ and $n = 5$. In the energy intervals, where this function is zero, the electron can move, otherwise the crystal is an insulator.

THE INVERSE OF A MATRIX. Because the columns of $A^{-1}$ are solutions of $Ax = e_i$ with

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

Cramer's rule together with the Lagrange expansion gives $A^{-1} = e_j \cdot A^{-1} e_i = e_j \cdot x_j = (-1)^{i+j} \det(A_{ji})/\det(A)$.

The matrix $B_{ij} = (-1)^{i+j} \det(A_{ji})$ is called the classical adjoint of $A$. NOTE the change $ij \rightarrow ji$. DON'T confuse the classical adjoint with the transpose $A^T$ which is sometimes also called the adjoint.